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NAVORD REPORT 2159

ON THE PROBLEM OF HEAT CONDUCTION IN CERTAIN QUASI-INFINITE TWO- AND THREE-DIMENSIONAL DOMAINS

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U. S. NAVAL ORDNANCE LABORATORY
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ON THE PROBLEM OF HEAT CONDUCTION IN CERTAIN QUASI-INFINITE TWO- AND THREE-DIMENSIONAL DOMAINS

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ABSTRACT: The object of the present paper is the derivation of the solutions of the problems in heat conduction in the following domains:

D_2 defined by $-\infty < x < \infty \; ; \; 0 \le y \le a$

D_2' defined by $0 < x < \infty \; ; \; 0 \le y \le a$.

D_3' defined by $-\infty < x < \infty \; ; \; -\infty < y < \infty \; ; \; 0 \le z \le a$

D_3'' defined by $-\infty < x < \infty \; ; \; 0 < y < \infty \; ; \; 0 \le z \le a$

D_3''' defined by $0 < x < \infty \; ; \; 0 < y < \infty \; ; \; 0 \le z \le a$

D_3^{IV} defined by $0 \le x \le a \; ; \; 0 \le y \le b \; ; \; -\infty < z < \infty$

D_3^{V} defined by $0 \le x \le a \; ; \; 0 \le y \le b \; ; \; 0 \le z < \infty$.

In the absence of a better term the above domains which extend to infinity in certain directions but remain bounded in other directions have been called quasi-infinite.

U. S. NAVAL ORDNANCE LABORATORY
White Oak, Maryland

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This report contains a method for the calculation of temperatures in certain quasi-infinite two and three dimensional domains. It is applicable to the solution of some types of heat transfer problems. The results are distributed to outside research laboratories for information, and for use in the solution of problems in heat conduction. This work was sponsored by the Office of Naval Research, project number NR-044-003, entitled "Numerical Analysis." Why?

EDWARD L. WOODYARD
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By direction

Part I. Heat conduction in the domain D.

Section I. Boundaries kept at prescribed temperatures.

It obviously suffices to consider the case when the temperature is prescribed on one of the boundaries, say $y=0$, the other boundary being kept at 0°C.

Accordingly the mathematical formulation of the problem is as follows:

$$T: \begin{cases} \left\{ \frac{\partial}{\partial t} - \kappa \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right\} T(x,y;t) = 0 & (1) \\ \lim_{t \to 0} T(x,y;t) = f(x,y) & (2) \\ T(x,0;t) = \varphi(x;t) & (3) \\ T(x,a;t) = 0 \ . & (4) \end{cases}$$

To solve the system T we put

$$T(x,y;t) = u(x,y;t) + v(x,y;t) \tag{5}$$

where $u(x,y;t)$ and $v(x,y;t)$ satisfy the differential equation (1) and the following initial and boundary conditions

$$\lim_{t \to 0} u(x,y;t) = 0 \tag{6}$$

$$u(x,0;t) = \varphi(x;t) \tag{7}$$

$$u(x,a;t) = 0 \tag{8}$$

$$\lim_{t \to 0} v(x,y;t) = f(x,y) \tag{9}$$

$$v(x,0;t) = 0 \tag{10}$$

$$v(x,a;t) = 0 \ . \tag{11}$$

Derivation of solution u(x,y;t)

The Laplace transform

$$u^*(x,y;p) = \mathcal{L}\{u(x,y;t)\} = \int_0^\infty e^{-pt} u(x,y;t)\,dt$$

must satisfy the system

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{p}{k} \right) u^*(x,y;p) = 0 \tag{12}$$

$$u^*(x,0;p) = \varphi^*(x;p) = \mathcal{L}\{\varphi(x;t)\} \tag{13}$$

$$u^*(x,a;p) = 0 \ . \tag{14}$$

The expression

$$u^*(x,y;p) = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^\infty \varphi^*(\xi;p) \cdot \frac{\sinh\beta(a-y)}{\sinh\beta a} \cdot \cos\alpha(x-\xi)\, d\xi \tag{15}$$

where $\beta = \sqrt{\frac{p}{k} + \alpha^2}$ is readily seen to satisfy the last three equations.

If we put

$$\int_0^\infty e^{-pt}\, \psi(y;t,\alpha)\, dt = \frac{\sinh\beta(a-y)}{\sinh\beta a} \tag{16}$$

then (15) yields:

$$u(x,y;t) = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^\infty \cos\alpha(x-\xi)\, d\xi \cdot \int_0^t \varphi(\xi;t-\tau)\, \psi(y;\tau,\alpha)\, d\tau \ . \tag{17}$$

In order to evaluate the function $\psi(y;t,\alpha)$ we associate with (16) the integral equation

$$\int_0^\infty e^{-pt}\, \Phi(y;t)\, dt = \frac{\sinh q(a-y)}{\sinh qa} \tag{18}$$

where $q = \sqrt{\frac{p}{k}}$. Since (18) must be an identity in p, we may replace p by $p + k\alpha^2$; then (18) becomes

$$\int_0^\infty e^{-pt}\, e^{-k\alpha^2 t}\, \Phi(y;t)\, dt = \frac{\sinh\beta(a-y)}{\sinh\beta a}$$

whence

$$\psi(y;t,\alpha) = e^{-k\alpha^2 t}\, \Phi(y;t) \ . \tag{19}$$

Substituting (19) in (17) and making use of the identity

$$\int_0^\infty e^{-k\alpha^2\tau} \cos\alpha(x-\xi)\,d\alpha = \frac{\sqrt{\pi}}{2\sqrt{k\tau}} \cdot e^{-\frac{(x-\xi)^2}{4k\tau}} \tag{20}$$

(See [1] , p. 31)*

we get

$$u(x,y,t) = \frac{1}{2\sqrt{\pi k}} \int_{-\infty}^{\infty} d\xi \int_0^t e^{-\frac{(x-\xi)^2}{4k\tau}} \tau^{-\frac{1}{2}} \varphi(\xi;t-\tau)\, \Phi(y;\tau)\,d\tau \,. \tag{21}$$

From (18), the Inversion Theorem yields

$$\Phi(y;t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} \cdot \frac{\sinh\mu(a-y)}{\sinh\mu a}\, d\lambda \tag{22}$$

where $\mu = \sqrt{\frac{\lambda}{k}}$, and σ is chosen so that the poles of the integrand are to the left of the line $\sigma - i\infty$, $\sigma + i\infty$.

(See [2] , p.71)

It may be verified that

$$|F(\lambda)| = \left|e^{\lambda t} \frac{\sinh\mu(a-y)}{\sinh\mu a}\right| < CR^{-k} \tag{23}$$

where $\lambda = Re^{i\theta}$, $-\pi \leq \theta \leq \pi$, $R > R_0$ where R_0 , C and k are constants and $k > 0$. Under these conditions, it is known that $\Phi(y;t)$ becomes equal to the sum of residues at the poles of $F(\lambda)$.

(See above reference p. 76)

We thus obtain

$$\Phi(y;t) = \frac{2\pi k}{a^2} \cdot \sum_{n=1}^{\infty} n \sin\frac{\pi n y}{a} \cdot e^{-\frac{k\pi^2 n^2 t}{a^2}} \,. \tag{24}$$

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*Numbers in square brackets refer to items listed in the bibliography.

In view of (24), (21) becomes:

$$u(x,y;t) = \frac{\sqrt{\pi k}}{a^2} \sum_{n=1}^{\infty} n \sin \frac{n\pi y}{a} \tag{25}$$

$$\cdot \int_{-\infty}^{\infty} d\xi \int_0^t \frac{e^{-\frac{(x-\xi)^2}{4k\tau}}}{\sqrt{\tau}} \varphi(\xi;t-\tau) \cdot e^{-\frac{kn^2\pi^2\tau}{a^2}} d\tau \ .$$

In the special case when φ is independent of x so that our problem becomes one-dimensional, bearing in mind the identity

$$\frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} d\xi = 1 \tag{26}$$

(This follows by an obvious transformation from the known identity $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$.) equation (25) becomes

$$u(y;t) = \frac{2k\pi}{a^2} \sum_{n=1}^{\infty} n \sin \frac{n\pi y}{a} \int_0^t \varphi(t-\tau)\, e^{-\frac{kn^2\pi^2\tau}{a^2}} d\tau \tag{27}$$

in agreement with the result given by Carslaw [1], p. 180

<u>Derivation of solution v(x,y;t)</u>

The Laplace transform $v^*(x,y;p)$ of v(x,y;t) must satisfy the following differential equation and boundary conditions:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{p}{k}\right) v^*(x,y;p) = -\frac{1}{k} f(x,y) \tag{28}$$

$$v^*(x,0;p) = v^*(x,a;p) = 0 \ . \tag{29}$$

In view of the identity

$$\Phi(x,y) = \frac{1}{\pi^2} \iint_{-\infty}^{\infty} d\xi\, d\eta \iint_0^{\infty} \Phi(\xi,\eta) \cos\alpha(x-\xi) \cos\beta(y-\eta)\, d\alpha\, d\beta \tag{30}$$

it follows that

$$v^*(x,y;p) = \frac{1}{\pi^2} \iint_{-\infty}^{\infty} \Phi(\xi,\eta)\, d\xi\, d\eta \iint_0^{\infty} \frac{\cos\alpha(x-\xi)\cos\beta(y-\eta)}{k(\alpha^2+\beta^2)+p}\, d\alpha\, d\beta \tag{31}$$

will satisfy (28) and (29), provided that

$$\begin{cases} \Phi(x, y+2na) = f(x,y) & 0 \le y \le a \\ \Phi(x, -y+2na) = -f(x,y) & \pm n = \pm 1, \pm 2, \pm 3, \dots \pm \infty . \end{cases} \tag{32}$$

From (31) it follows that

$$v(x,y,t) = \frac{1}{\pi^2} \iint_{-\infty}^{\infty} \Phi(\xi,\eta)\, d\xi\, d\eta \iint_0^{\infty} e^{-k(\alpha^2+\beta^2)t} \cdot \cos \alpha(x-\xi) \cos \beta(y-\eta)\, d\alpha\, d\beta . \tag{33}$$

In view of (32) and (20), (33) becomes

$$v(x,y;t) = \frac{1}{4\pi kt} \int_{-\infty}^{\infty} f(\xi,\eta)\, e^{-\frac{(x-\xi)^2}{4kt}}\, d\xi \cdot \int_0^a \left\{ \sum_{n=-\infty}^{\infty} \left[e^{-\frac{(y-\eta-2na)^2}{4kt}} - e^{-\frac{(y+\eta-2na)^2}{4kt}} \right] \right\} d\eta . \tag{34}$$

From the identity

$$\sum_{n=-\infty}^{\infty} e^{2n\pi\nu i - n^2\pi^2\tau} \equiv \sum_{n=-\infty}^{\infty} \frac{e^{-(\nu+n)^2/\tau}}{\sqrt{\pi\tau}} \tag{35}$$

by some obvious transformations we obtain the identities:

$$\sum_{-\infty}^{\infty} e^{-\frac{(y+\eta-2na)^2}{4kt}} = \frac{\sqrt{\pi kt}}{a} \left\{ 1 + 2\sum_{n=1}^{\infty} e^{-\frac{kn^2\pi^2 t}{a^2}} \cos \frac{n\pi}{a}(y+\eta) \right\} \tag{36}$$

$$\sum_{-\infty}^{\infty} e^{-\frac{(y-\eta-2na)^2}{4kt}} = \frac{\sqrt{\pi kt}}{a} \left\{ 1 + 2\sum_{n=1}^{\infty} e^{-\frac{kn^2\pi^2 t}{a^2}} \cos \frac{n\pi}{a}(y-\eta) \right\} . \tag{36'}$$

In view of (36) and (36'), (34) becomes

$$v(x,y;t) = \frac{1}{a\sqrt{\pi kt}} \sum_{n=1}^{\infty} \sin \frac{n\pi y}{a} \cdot e^{-\frac{kn^2\pi^2 t}{a^2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}}\, d\xi \int_0^a f(\xi,\eta) \sin \frac{n\pi\eta}{a}\, d\eta . \tag{37}$$

In the special case when f(x,y) becomes a function of y only (37) becomes in view of (26)

$$v(y;t) = \frac{2}{a}\sum_{n=1}^{\infty} \sin\frac{n\pi y}{a}\int_0^a \sin\frac{n\pi\eta}{a}\cdot e^{-\frac{k n^2\pi^2 t}{a^2}} f(\eta)\,d\eta \tag{38}$$

in agreement with the result given by Carslaw [1], p. 180 .

In conclusion, the final solution of the system T is given by (5) in conjunction with (25) and (37).

Section 2.

Boundary y=0 kept at temperature $\varphi_1(x;t)$; temperature gradient $\varphi_2(x;t)$ prescribed on y=a. Initial temperature f(x,y).

In this case we put

$$T(x,y;t) = u_1(x,y;t) + u_2(x,y;t) + v(x,y;t) \tag{39}$$

where u_1, u_2 and v satisfy the differential equation (1) and where

$$\lim_{t\to 0} u_1(x,y;t) = 0 \tag{40}$$

$$u_1(x,0;t) = \varphi_1(x;t) \tag{41}$$

$$\frac{\partial}{\partial y} u_1(x,y;t) = 0 \quad \text{for } y=a \tag{42}$$

$$\lim_{t\to 0} u_2(x,y;t) = 0 \tag{43}$$

$$u_2(x,0;t) = 0 \tag{44}$$

$$\frac{\partial}{\partial y} u_2(x,y;t) = \varphi_2(x;t) \quad \text{for } y=a \tag{45}$$

$$\lim_{t\to 0} v(x,y;t) = f(x,y) \tag{46}$$

$$v(x,0;t) = 0 \tag{47}$$

$$\frac{\partial}{\partial y} v(x,y;t) = 0 \quad \text{for } y=a . \tag{48}$$

Derivation of solution $u_1(x,y;t)$

The Laplace transform $u_1^*(x,y;p)$ of $u_1(x,y;t)$ is given by

$$u_1^*(x,y;p) = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^{\infty} \varphi_1^*(\xi;p)\,\frac{\cosh\beta(a-y)}{\cosh\beta a}\cdot\cos\alpha(x-\xi)\,d\xi \tag{49}$$

where $\beta = \sqrt{\frac{p}{k}+\alpha^2}$. From (49) it follows that

$$u_1(x,y;t) = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^{\infty}\cos\alpha(x-\xi)\,d\xi\int_0^t \varphi_1(\xi;t-\tau)\,\psi(y;\tau,\alpha)\,d\tau \tag{50}$$

where

$$\int_0^\infty e^{-pt}\,\psi(y;t,\alpha)\,dt = \frac{\cosh\beta(a-y)}{\cosh\beta a}.$$

By analogy with the developments in Section 1 we have

$$\psi(y;t,\alpha) = e^{-k\alpha^2 t}\,\Phi(y;t) \tag{51}$$

where

$$\int_0^\infty e^{-pt}\,\Phi(y;t)\,dt = \frac{\cosh q(a-y)}{\cosh qa}$$

where $q = \sqrt{\frac{p}{k}}$. Moreover, as in Section 1, the value of $\Phi(y;t)$ is obtained as the sum of the residues of

$$F(p) = e^{pt}\,\frac{\cosh q(a-y)}{\cosh qa}$$

where $q = \sqrt{\frac{p}{k}}$. Thus we ultimately obtain

$$\Phi(y;t) = \frac{k\pi}{a^2}\sum_{m=0}^{\infty}(2m+1)\sin\frac{(2m+1)\pi y}{2a}\,e^{-\frac{k(2m+1)^2\pi^2 t}{4a^2}}. \tag{52}$$

In view of (51), (52) and (20), (50) becomes

$$u_1(x,y;t) = \frac{\sqrt{\pi k}}{2a^2}\sum_{m=0}^{\infty}(2m+1)\sin\frac{(2m+1)\pi y}{2a}\int_{-\infty}^{\infty}d\xi\int_0^t \varphi_1(\xi;t-\tau)\,\tau^{-\frac{1}{2}}\,e^{-\frac{k(2m+1)^2\pi^2\tau}{4a^2}}\,d\tau. \tag{53}$$

In the special case where the function φ_1 is independent of x, equation (53) reduces to

$$u_1(y;t) = \frac{k\pi}{a^2}\sum_{m=0}^{\infty}(2m+1)\sin\frac{(2m+1)\pi y}{2a}$$

$$\int_0^t \varphi_1(t-\tau)\, e^{-\frac{k(2m+1)^2\pi^2\tau}{4a^2}}\, d\tau \;. \tag{54}$$

Derivation of solution $u_2(x,y;t)$

The counterpart of (50) is

$$u_2^*(x,y;p) = \frac{1}{\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} \varphi_2^*(\xi;p)\, \frac{\sinh \beta y}{\beta \cosh \beta a}\cdot \cos\alpha(x-\xi)\, d\xi \tag{55}$$

where $\beta = \sqrt{\frac{p}{k}+\alpha^2}$, whence

$$u_2(x,y;t) = \frac{1}{\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty}\cos\alpha(x-\xi)\, d\xi \int_0^t \varphi_2(\xi;t-\tau)\, \psi(y;\tau,\alpha)\, d\tau \tag{56}$$

where, now

$$\int_0^{\infty} e^{-pt}\,\psi(y;t,\alpha)\, dt = \frac{\sinh \beta y}{\beta\cosh\beta a} \;.$$

Proceeding as in the previous cases, we have

$$\psi(y;t,\alpha) = e^{-k\alpha^2 t}\, \Phi_2(y;t) \tag{57}$$

where the expression for $\Phi(y;t)$ is obtained as the sum of residues of

$$F(p) = e^{pt}\,\frac{\sinh q y}{q\cosh q a}$$

at its poles. Thus we finally get

$$\Phi(y;t) = \frac{2k}{a}\sum_{m=0}^{\infty}(-1)^m\, e^{-\frac{(2m+1)^2\pi^2 kt}{4a^2}}\, \sin\frac{(2m+1)\pi y}{2a} \;. \tag{58}$$

In view of (58), (57), and (20), (56) becomes

$$u_2(x,y;t) = \frac{2\sqrt{k}}{a\sqrt{\pi}}\sum_{m=0}^{\infty}(-1)^m \sin\frac{(2m+1)\pi y}{2a}$$

$$\int_{-\infty}^{\infty} d\xi \int_0^t \varphi_2(\xi;t-\tau)\cdot e^{-\frac{(x-\xi)^2}{4k\tau}}\cdot e^{-\frac{(2m+1)^2\pi^2 k\tau}{4a^2}}\cdot \tau^{-\frac{1}{2}}\, d\tau \;. \tag{59}$$

In the special case where φ_2 is independent of x equation (59) in view of (26) becomes

$$u_2(y;t) = \frac{4k}{a}\sum_{n=0}^{\infty}(-1)^n \sin\frac{(2n+1)\pi y}{2a} \cdot \int_0^t \varphi_2(t-\tau)\cdot e^{-\frac{(2n+1)^2\pi^2 k\tau}{4a^2}}\, d\tau \;. \tag{60}$$

Derivation of solution $v(x,y;t)$

The solution $v(x,y;t)$ is a y identical with the solution of the problem of heat conduction in a slab of thickness 2a, whose bounding planes are kept at 0^oC, initially at a temperature $\phi(x,y)$ defined by

$$\phi(x,y) = f(x,y) \qquad \text{for} \quad 0 < y < a$$
$$\phi(x,2a-y) = f(x,y) \qquad \text{for} \quad a < y < 2a \;.$$

This leads to

$$v(x,y;t) = \frac{1}{a\sqrt{\pi k t}}\sum_{n=0}^{\infty}\sin\frac{(2n+1)\pi y}{2a}\cdot e^{-\frac{(2n+1)^2\pi^2 kt}{4a^2}} \cdot\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}}\,d\xi \int_0^a f(\xi,\eta)\sin\frac{(2n+1)\pi\eta}{2a}\,d\eta \;. \tag{61}$$

When the initial temperature is independent of x, the last equation with the aid of (26) becomes

$$v(y;t) = \frac{2}{a}\sum_{n=0}^{\infty}\sin\frac{(2n+1)\pi y}{2a}\cdot e^{-\frac{(2n+1)^2\pi^2 kt}{4a^2}}\int_0^a f(\eta)\sin\frac{(2n+1)\pi\eta}{2a}\,d\eta \;. \tag{62}$$

The result in (62) is not given by Carslaw; it may however be derived from his solution "u" [p. 68] and it is found to agree with (62).

In conclusion, the final solution of our present problem is given by (39) in conjunction with (53), (59), and (61). When the boundary y=a is impervious to heat, $u_2(x,y;t) = 0$ and $T(x,y;t) = u_1(x,y;t) + u_2(x,y;t)$.

Section 3.

Radiation at the boundary y=0 into a medium at temperature $\varphi_1(x;t)$; boundary y=a kept at the temperature $\varphi_2(x;t)$. Initial temperature f(x,y).

In this case we put

$$T(x,y;t) = u_1(x,y;t) + u_2(x,y;t) + v_1(x,y;t) + v_2(x,y;t) \tag{63}$$

where u_1 , u_2 , v_1 and v_2 are solutions of (1) satisfying the following initial and boundary conditions:

$$\lim_{t\to 0} u_1(x,y;t) = 0$$

$$\left(\frac{\partial}{\partial y} - h\right) u_1(x,y;t) = -h\varphi_1(x;t) \qquad \text{for } y=0$$

$$u_1(x,a;t) = 0$$

$$\lim_{t\to 0} u_2(x,y;t) = 0$$

$$\left(\frac{\partial}{\partial y} - h\right) u_2(x,y;t) = 0 \qquad \text{for } y=0$$

$$u_2(x,a;t) = \varphi_2(x;t)$$

$$\lim_{t\to 0} v_1(x,y;t) = f(x,y)$$

$$\frac{\partial}{\partial y} v_1(x,y;t) = 0 \qquad \text{for } y=0$$

$$v_1(x,a;t) = 0$$

$$\lim_{t\to 0} v_2(x,y;t) = 0$$

$$\left(\frac{\partial}{\partial y} - h\right) v_2(x,y;t) = h v_1(x,0;t) \qquad \text{for } y=0$$

$$v_2(x,a;t) = 0 \ .$$

In the special case where $\varphi_1(x;t) \equiv 0$, i.e. when radiation takes place at y=0 into a medium at 0°C, it is clear that $u_1(x,y;t) \equiv 0$. Nevertheless, it will be noted that the initial and boundary conditions satisfied by $v_2(x,y;t)$ are similar to those satisfied by $u_1(x,y;t)$. Thus the formal solution $u_1(x,y;t)$ for $\varphi_1(x;t) \neq 0$ - is necessary for obtaining $v_2(x,y;t)$ even in the special case where radiation at y=0 takes place into a medium at 0°C.

Derivation of solution $u_1(x,y;t)$

The Laplace transform $u_1^*(x,y;p)$ is easily obtained in the form

$$u_1^*(x,y;p) = \frac{h}{\pi}\int_0^\infty d\alpha \int_{-\infty}^{\infty} \varphi_1^*(\xi;p) \frac{\sinh\beta(y-a)}{\beta\cosh\beta a + h\sinh\beta a} \cdot \cos\alpha(x-\xi)\, d\xi$$

where $\beta = \sqrt{\frac{p}{k}+\alpha^2}$, whence

$$u_1(x,y;t) = \frac{h}{\pi}\int_{-\infty}^{\infty} d\xi \int_0^t \varphi_1(\xi;t-\tau)\,\Phi(y;\tau)\,d\tau \cdot \int_0^\infty e^{-k\alpha^2\tau}\cos\alpha(x-\xi)\,d\alpha \tag{64}$$

$$= \frac{h}{2\sqrt{\pi k}}\int_{-\infty}^{\infty} d\xi \int_0^t \varphi_1(\xi;t-\tau)\,\Phi(y;\tau)\cdot e^{-\frac{(x-\xi)^2}{4k\tau}}\,\tau^{-\frac{1}{2}}\,d\tau$$

where $\Phi(y;t)$ is equal to the sum of the residues of

$$F(p) = e^{pt} \cdot \frac{\sinh q(y-a)}{q\cosh qa + h\sinh qa}$$

where $q = \sqrt{\frac{p}{k}}$, at its poles. We ultimately obtain

$$\Phi(y;t) = 2hk \cdot \sum_{m=1}^{\infty} \frac{\zeta_m \sin(1-\frac{y}{a})\zeta_m \cdot e^{-kt\zeta_m^2/a^2}}{[(1+ah)ah+\zeta_m^2]\cos\zeta_m} \tag{65}$$

where the summation extends over the roots of the transcendental equation

$$\zeta + ah\tan\zeta = 0 . \tag{66}$$

In view of (65), (64) becomes

$$u_1(x,y;t) = \frac{h^2\sqrt{k}}{\sqrt{\pi}} \cdot \sum_{m=1}^{\infty} \frac{\zeta_m \sin(1-\frac{y}{a})\zeta_m}{[(1+ah)ah+\zeta_m^2]\cos\zeta_m} \cdot \int_{-\infty}^{\infty} d\xi \int_0^t \varphi_1(\xi;t-\tau)\cdot e^{-\frac{(x-\xi)^2}{4k\tau}}\, e^{-\frac{k\tau\zeta_m^2}{a^2}}\,\tau^{-\frac{1}{2}}\,d\tau \tag{67}$$

where the summation extends over the roots of (66). As in the previous cases, when the function φ_1 is independent of x, (67) reduces with the aid of (26) to

$$u_1(y;t) = 2h^2 k \sum_{n=1}^{\infty} \frac{\zeta_n \sin\left(1-\frac{y}{a}\right)\zeta_n}{\left[(1+ah)ah+\zeta_n^2\right]\sin\zeta_n} \cdot \int_0^t \varphi_1(t-\tau)\, e^{-\frac{k\tau\zeta_n^2}{a^2}}\, d\tau \,, \tag{68}$$

the summation being extended over the roots of (66).

Derivation of solution $u_2(x,y;t)$

The Laplace transform $u_2^*(x,y;p)$ is obtained in the form

$$u_2^*(x,y;p) = \frac{1}{\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} \varphi_2^*(\xi,p)\, \frac{\beta\cosh\beta y + h\sinh\beta y}{\beta\cosh\beta a + h\sinh\beta a}\, \cos\alpha(x-\xi)\, d\xi$$

where $\beta = \sqrt{\frac{p}{k}+\alpha^2}$. It is clear that $u_2(x,y;t)$ is given by a formula similar to (64), except that $\varphi_1(x;t)$ is replaced by $\frac{1}{k}\varphi_2(x,t)$ and $\Phi(y;t)$ has for its Laplace transform the expression

$$\frac{q\cosh qy + h\sinh qy}{q\cosh qa + h\sinh qa}$$

where $q = \sqrt{\frac{p}{k}}$. As before $\Phi(y,t)$ is given by the sum of residues of the above expression multiplied by the factor e^{pt}. Accordingly we get

$$\Phi(y;t) = -2hk \sum_{n=1}^{\infty} \frac{\zeta_n \cos\frac{y}{a}\zeta_n + ah\sin\frac{y}{a}\zeta_n}{\left[(1+ah)ah+\zeta_n^2\right]\cos\zeta_n}\, \zeta_n\, e^{-\frac{kt\zeta_n^2}{a^2}}$$

where the summation extends over the roots of (66). With the aid of the last equation the desired expression for $u_2(x,y;t)$ becomes

$$u_2(x,y;t) = -\frac{h\sqrt{k}}{a\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{\zeta_n \cos\frac{y}{a}\zeta_n + ah\sin\frac{y}{a}\zeta_n}{\left[(1+ah)ah+\zeta_n^2\right]\cos\zeta_n}\, \zeta_n\, e^{-\frac{k\tau\zeta_n^2}{a^2}} \cdot \int_{-\infty}^{\infty} d\xi \int_0^t \varphi_2(\xi,t-\tau)\, e^{-\frac{(x-\xi)^2}{4k\tau}} \cdot e^{-\frac{k\tau\zeta_n^2}{a^2}}\, \tau^{-\frac{1}{2}}\, d\tau \,, \tag{69}$$

the summation extending over the roots of (66).

Derivation of solution $v_1(x,y;t)$

By analogy with the developments in Section 1, the Laplace transform $\bar{v}_1(x,y;p)$ is given by (31) where $\phi(x;y)$ satisfies the conditions

$$\left\{ \begin{array}{l} \phi(x, \pm y + 4ma) = f(x,y) \\ \phi(x, \pm y + 4ma + 2a) = -f(x,y) \end{array} \right\} \quad \begin{array}{l} 0 < y < a \\ m = \pm 1, \pm 2, \pm 3, \dots \pm \infty . \end{array} \tag{70}$$

From (31) and (70) it follows that

$$\begin{aligned} v_1(x,y;t) = \frac{1}{4kt} \int_{-\infty}^{\infty} F(\xi,\eta) \cdot e^{-\frac{(x-\xi)^2}{4kt}} \, d\xi \\ \cdot \int_0^a \Bigg\{ \sum_{m=-\infty}^{\infty} \left[e^{-\frac{(y+\eta-4ma)^2}{4kt}} + e^{-\frac{(y-\eta-4ma)^2}{4kt}} \right] \\ - \sum_{m=-\infty}^{\infty} \left[e^{-\frac{(y+\eta-4ma-2a)^2}{4kt}} + e^{-\frac{(y-\eta-4ma-2a)^2}{4kt}} \right] \Bigg\} d\eta . \end{aligned} \tag{71}$$

Making use of the identities (36) and (36'), (71) becomes after some obvious transformations

$$\begin{aligned} v_1(x,y;t) = \frac{1}{a\sqrt{\pi k t}} \cdot \sum_{n=0}^{\infty} \cos \frac{(2n+1)\pi y}{2a} \cdot e^{-\frac{(2n+1)^2 k \pi^2 t}{4a^2}} \\ \cdot \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} d\xi \int_0^a f(\xi,\eta) \cos \frac{(2n+1)\pi\eta}{2a} \, d\eta . \end{aligned} \tag{72}$$

Derivation of solution $v_2(x,y;t)$

Comparing the initial and boundary conditions satisfied by $v_2(x,y;t)$ with those satisfied by $u_1(x,y;t)$ we conclude that the expression of $v_2(x,y;t)$ may be obtained from (64) by replacing $\varphi_1(x,t)$ by $-v_1(x,0;t)$. Thus

$$\begin{aligned} v_2(x,y;t) = -\frac{h^2\sqrt{k}}{\sqrt{\pi}} \sum_{m=1}^{\infty} \frac{\zeta_m \sin\left(1-\frac{y}{a}\right)\zeta_m}{\left[(1+ah)ah + \zeta_m^2\right] \cos \zeta_m} \\ \cdot \int_{-\infty}^{\infty} d\xi \int_0^t v_1(\xi,0;t-\tau) \cdot e^{-\frac{(x-\xi)^2}{4k\tau}} e^{-\frac{k\zeta_m^2 \tau}{a^2}} \tau^{-\frac{1}{2}} \, d\tau \end{aligned} \tag{73}$$

where $v_1(x, 0; t)$ is obtained from (72).

In conclusion the desired solution $T(x, y; t)$ is obtained from (63) in conjunction with (67), (69), (72), and (73).

Section 4.

Boundary y=0 radiating into a medium at temperature $\varphi_1(x, t)$; temperature gradient $\varphi_2(x, t)$ prescribed on y=a. Initial temperature $f(x, y)$.

In this case we put

$$T(x, y; t) = u_1(x, y; t) + u_2(x, y; t) + v_1(x, y; t) + v_2(x, y; t) \quad (74)$$

where u_1 , u_2 , v_1 and v_2 satisfy the differential equation (1), and the following initial and boundary conditions

$$\lim_{t \to 0} u_1(x, y; t) = 0$$

$$\left(\frac{\partial}{\partial y} - h\right) u_1(x, y; t) = -h\varphi_1(x; t) \qquad \text{for } y = 0$$

$$\frac{\partial}{\partial y} u_1(x, y; t) = 0 \qquad \text{for } y = a$$

$$\lim_{t \to 0} u_2(x, y; t) = 0$$

$$\left(\frac{\partial}{\partial y} - h\right) u_2(x, y; t) = 0 \qquad \text{for } y = 0$$

$$\frac{\partial}{\partial y} u_2(x, y; t) = \varphi_2(x; t) \qquad \text{for } y = a$$

$$\lim_{t \to 0} v_1(x, y; t) = f(x, y)$$

$$\frac{\partial}{\partial y} v_1(x, y; t) = 0 \qquad \text{for } y = 0 \text{ and } y = a$$

$$\lim_{t \to 0} v_2(x, y; t) = 0$$

$$\frac{\partial}{\partial y} v_2(x, y; t) = h\{v_2(x, y; t) + v_1(x, y; t)\} \qquad \text{for } y = 0$$

$$\frac{\partial}{\partial y} v_2(x, y; t) = 0 \qquad \text{for } y = a .$$

Derivation of solution $u_1(x,y;t)$

The expression of the Laplace transform is

$$u_1^*(x,y;p) = \frac{k}{\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} \varphi_1^*(\xi;p)\, \psi^*(y;p,\alpha)\, \cos\alpha(x-\xi)\, d\xi \tag{75}$$

where

$$\psi^*(y;p,\alpha) = \frac{\cosh\beta(y-a)}{\beta\sinh\beta a + h\cosh\beta a} \tag{76}$$

with $\beta = \sqrt{\frac{p}{k} + \alpha^2}$. By analogy with the developments in the previous sections, we have

$$u_1(x,y;t) = \frac{k}{\pi}\int_{-\infty}^{\infty} d\xi \int_0^t d\tau \int_0^{\infty} \varphi_1(\xi;t-\tau)\, \Phi(y;\tau)\, e^{-k\alpha^2\tau} \cos\alpha(x-\xi)\, d\alpha$$

where the expression for $\Phi(y;t)$ is given by the sum of the residues of $e^{pt}\Phi^*(y,p)$ where $\Phi^*(y;p)$ is obtained from $\psi^*(y;p,\alpha)$ by replacing β by $q = \sqrt{\frac{p}{k}}$.

In this manner, we finally get

$$u_1(x,y;t) = \frac{k^2\sqrt{k}}{\sqrt{\pi}} \cdot \sum_{n=1}^{\infty} \frac{\zeta_n\, e^{-\frac{k t\zeta_n^2}{a^2}} \cos\zeta_n\left(1-\frac{y}{a}\right)}{\left\{(1+ah)ah + \zeta_n^2\right\}\sin\zeta_n}$$

$$\cdot\int_{-\infty}^{\infty} d\xi \int_0^t e^{-\frac{k\tau\zeta_n^2}{a^2}} \cdot e^{-\frac{(x-\xi)^2}{4k\tau}}\, \varphi_1(\xi;t-\tau)\cdot\tau^{-\frac{1}{2}}\, d\tau \tag{77}$$

where the summation extends over the roots of

$$\zeta\tan\zeta = ah\ . \tag{78}$$

Derivation of solution $u_2(x,y;t)$

The expression of the Laplace transform is

$$u_2^*(x,y;p) = \frac{1}{\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} \varphi_2^*(\xi;p)\, \psi^*(y;p,\alpha)\, \cos\alpha(x-\xi)\, d\xi \tag{79}$$

where

$$\psi^*(y;p,\alpha) = \frac{\beta\cosh\beta y + h\sinh\beta y}{\beta^2\sinh\beta a + h\beta\cosh\beta a}\ . \tag{80}$$

Proceeding as above, we finally get

$$u_1(x, y; t) = \frac{\sqrt{k}}{a\sqrt{\pi}} \cdot \sum_{n=1}^{\infty} \frac{\zeta_n(\zeta_n \cos y\zeta_n + ah \sin y\zeta_n)}{(2 + ah)\zeta_n \sin \zeta_n + (ah + \zeta_n^2) \cos \zeta_n}$$

$$\cdot \int_{-\infty}^{\infty} d\xi \int_0^t e^{-\frac{k\tau\zeta_n^2}{a^2}} \cdot e^{-\frac{(x-\xi)^2}{4k\tau}} \varphi_1(\xi; t-\tau) \cdot \tau^{-\frac{1}{2}} \, d\tau \, , \tag{81}$$

the summation being extended ov-r the roots of (78).

<u>Derivation of solution $v_1(x,y;t)$</u>

By analogy with the developments in Section 1, the expression for $v_1(x, y; t)$ is given by (33) where $\phi(x, y)$ satisfies the conditions

$$\left\{ \begin{aligned} &\phi(x, y + 2ma) = f(x,y) \\ &\phi(x, y + 2ma + a) = f(x, a-y) \end{aligned} \right\} \qquad \begin{aligned} &0 \leq y \leq a \\ &m = 0, \pm 1, \pm 2, \ldots \pm \infty \, . \end{aligned}$$

With the aid of these conditions, (33) yields

$$v_1(x, y; t) = \frac{1}{4\pi k t} \int_{-\infty}^{\infty} f(\xi, \eta) \, e^{-\frac{(x-\xi)^2}{4kt}} \, d\xi$$

$$\int_0^a \left\{ \sum_{m=-\infty}^{\infty} \left[e^{-\frac{(y+\eta-2ma)^2}{4kt}} + e^{-\frac{(y-\eta-2ma)^2}{4kt}} \right] \right\} d\eta \, .$$

In view of (36) and (36'), the last equation becomes

$$v_1(x, y; t) = \frac{1}{2a\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} d\xi \int_0^a f(\xi, \eta) \, d\eta$$

$$+ \frac{1}{a\sqrt{\pi k t}} \sum_{n=1}^{\infty} e^{-\frac{k n^2 \pi^2 t}{a^2}} \cos \frac{n\pi y}{a} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} d\xi \int_0^a f(\xi, \eta) \cos \frac{n\pi\eta}{a} \, d\eta \, . \tag{82}$$

In the case where the initial temperature distribution is a function of y only, the last equation becomes

$$v_1(y; t) = \frac{1}{a} \int_0^a f(\eta) \, d\eta + \frac{2}{a} \sum_{n=1}^{\infty} e^{-\frac{k n^2 \pi^2 t}{a^2}} \cos \frac{n\pi y}{a} \int_0^a f(\eta) \cos \frac{n\pi\eta}{a} \, d\eta \, . \tag{83}$$

Derivation of solution $v_2(x,y;t)$

From the initial and boundary conditions satisfied by $v_1(x, y; t)$ and $u_1(x, y; t)$ it is clear that the expression for $v_2(x, y; t)$ may be obtained from that of $u_1(x, y; t)$ by replacing $\varphi_1(x;t)$ by $-v_1(x, 0; t)$.

Thus

$$v_2(x,y;t) = \frac{h\sqrt{k}}{a\sqrt{\pi}} \cdot \sum_{n=1}^{\infty} \frac{\zeta_n e^{-\frac{kt\zeta_n^2}{a^2}} \cos \zeta_n (1-\frac{y}{a})}{\{(1+ah)ah + \zeta_n^2\}\sin\zeta_n} \cdot \int_{-\infty}^{\infty} d\xi \int_0^t e^{-\frac{k\tau\zeta_n^2}{a^2}} \cdot e^{-\frac{(x-\xi)^2}{4k\tau}} v_1(\xi, 0; t-\tau) \cdot \tau^{-\frac{1}{2}} d\tau \tag{84}$$

where $v_1(x, 0; t)$ is obtained from (82) and where the summation extends over the roots of (78).

Section 5.

Boundary $y=0$ radiating into a medium at temperature $\varphi_1(x;t)$; boundary $y=a$ radiating into a medium at temperature $\varphi_2(x;t)$.

In this case we put

$$T(x,y;t) = u_1(x,y;t) + u_2(x,y;t) + v_1(x,y;t) + v_2(x,y;t) \tag{85}$$

where u_1, u_2, v_1 and v_2 satisfy the differential equation (1) and the following initial and boundary conditions

$$\lim_{t\to 0} u_1(x,y;t) = 0$$

$$\left(\frac{\partial}{\partial y} - h_1\right) u_1(x,y;t) = -h_1\varphi_1(x;t) \qquad \text{for } y=0$$

$$\left(\frac{\partial}{\partial y} - h_2\right) u_1(x,y;t) = 0 \qquad \text{for } y=a.$$

$$\lim_{t\to 0} u_2(x,y;t) = 0$$

$$\left(\frac{\partial}{\partial y} - h_1\right) u_2(x,y;t) = 0 \qquad \text{for } y=0$$

$$\left(\frac{\partial}{\partial y} - h_2\right) u_2(x,y;t) = -h_2\varphi_2(x;t) \qquad \text{for } y=a$$

$$\lim_{t \to 0} v_1(x, y; t) = f(x, y)$$

$$\frac{\partial v_1}{\partial y} = 0 \qquad \text{for } y = 0 \text{ and } y = a$$

$$\lim_{t \to 0} v_2(x, y; t) = 0$$

$$\frac{\partial}{\partial y} v_2(x, y; t) = h_1 \{ v_2(x, y; t) + v_1(x, y; t) \} \quad \text{for } y = 0$$

$$\frac{\partial}{\partial y} v_2(x, y; t) = h_2 \{ v_2(x, y; t) + v_1(x, y; t) \} \quad \text{for } y = a \ .$$

In the above h_2 must be put equal to $-h_1$ for reasons discussed at the end of this section. It was convenient to formulate the boundary condition at y=a in terms of h_2 rather than $-h_1$ in order to be able subsequently to obtain the solution of two related problems by putting $h_2 = \infty$ or $h_2 = 0$.

<u>Derivation of solution $u_1(x,y;t)$</u>

Starting with the Laplace transform

$$u_1^*(x, y; p) = \frac{h_1}{\pi} \int_0^\infty d\alpha \int_{-\infty}^\infty \varphi_1^*(\xi; p)\, \psi^*(y; p, \alpha) \cos \alpha (x - \xi)\, d\xi$$

where

$$\psi^*(y; p, \alpha) = \frac{\beta \cosh \beta (y-a) + h_2 \sinh \beta (y-a)}{(\beta^2 - h_1 h_2) \sinh \beta a + (h_1 - h_2) \beta \cosh \beta a}$$

(with $\beta = \sqrt{\frac{p}{k} + \alpha^2}$) and proceeding as in Section 4 we ultimately get

$$u_1(x, y; t) = \frac{-h_1 \sqrt{k}}{a \sqrt{\pi}} \sum_{m=1}^{\infty} \frac{\zeta_m \{ a h_2 \sin (1 - \frac{y}{a}) \zeta_m - \zeta_m \cos (1 - \frac{y}{a}) \zeta_m \}}{\{ 2 + a(h_1 - h_1) \} \zeta_m \sin \zeta_m + \{ a^2 h_1 h_2 + a(h_2 - h_1) + \zeta_m^2 \} \cos \zeta_m} \tag{86}$$

$$\cdot \int_{-\infty}^{\infty} d\xi \int_0^t e^{-\frac{k\tau\zeta_m^2}{a^2}} \cdot e^{-\frac{(x-\xi)^2}{4k\tau}} \varphi_1(\xi; t-\tau) \cdot \tau^{-\frac{1}{2}}\, d\tau$$

where the summation extends over the roots of

$$(a^2 h_1 h_2 + \zeta^2)\tan\zeta - a(h_1 - h_2)\zeta = 0 \quad . \tag{87}$$

It is readily seen that if in (86) we put $h_2 = \infty$, we obtain the formula (67), the summation extending over the roots of (66), which results from the substitution $h_2 = \infty$ in (87). If in (86) and (87) we put $h_2 = 0$, we obtain the formula (77). The summation extending over the roots of (78) which results from the substitution $h_2 = 0$ in (87).

Derivation of solution $u_2(x,y;t)$

Starting with

$$u_2^*(x,y;p) = \frac{h_2}{\pi}\int_0^\infty d\alpha \int_{-\infty}^\infty \varphi_2^*(\xi;p)\, \psi^*(y;p,\alpha) \cos\alpha(x-\xi)\, d\xi \quad .$$

where

$$\psi^*(y;p,\alpha) = \frac{\beta\cosh\beta y + h_1\sinh\beta y}{(\beta^2 - h_1 h_2)\sinh\beta a + (h_1 - h_2)\beta\cosh\beta a}$$

with $\beta = \sqrt{\frac{p}{k} + \alpha^2}$, we ultimately get

$$u_2(x,y;t) = \frac{h_2\sqrt{k}}{a\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{\zeta_n\left\{\zeta_n\cos\frac{y}{a}\zeta_n + a h_1\sin\frac{y}{a}\zeta_n\right\}}{\{2 + a(h_1 - h_2)\}\zeta_n\sin\zeta_n + \{a^2 h_1 h_2 + a(h_1 - h_2) + \zeta_n^2\}\cos\zeta_n} \tag{88}$$

$$\cdot \int_{-\infty}^{\infty} d\xi \int_0^t e^{-\frac{k\tau\zeta_n^2}{a^2}} \cdot e^{-\frac{(x-\xi)^2}{4k\tau}} \varphi_2(\xi;t-\tau)\cdot\tau^{-\frac{1}{2}}\, d\tau \quad ,$$

the summation extending over the roots of (87).

Derivation of solution $v_1(x,y;t)$

The expression for $v_1(x,y;t)$ is obviously identical with that of Section 4 and is therefore given by (82).

Derivation of solution $v_2(x,y;t)$

From the boundary conditions satisfied by $v_2(x,y;t)$ it is clear that

$$v_2(x,y;t) = \bar{v}_1(x,y;t) + \bar{u}_2(x,y;t) \tag{89}$$

where $\bar{u}_1(x, y; t)$ is obtained from (86) by replacing $\varphi_1(x; t)$ by $-v_1(x, 0; t)$ and $\bar{u}_2(x, y; t)$ is obtained from (88) by replacing $\varphi_2(x; t)$ by $-v_1(x, a; t)$ where $v_1(x, 0; t)$ and $v_1(x, a; t)$ are obtained from (88).

In conclusion it should be noted that since the condition of radiation into a medium of prescribed temperature is $\frac{\partial T}{\partial n} = h(T - T_0)$ where h is positive and $\frac{\partial}{\partial n}$ denotes differentiation along the inwardly drawn normal, it follows that in the above developments we must put $h_2 = -h_1$.

Part II. <u>Heat Conduction in the Domain D_2'</u>

As in Part I, the general solution of the differential equation of heat conduction which reduces to a prescribed function $f(x, y)$ for t=0 and satisfies boundary conditions of the type

$$T(\bar{P}, t) = \varphi(\bar{P}, t)$$

$$\frac{\partial}{\partial n} T(\bar{P}, t) = \varphi(\bar{P}, t)$$

$$\left(\frac{\partial}{\partial n} - h\right) T(\bar{P}, t) = -h\varphi(\bar{P}, t)$$

where $\bar{P}$ denotes a point on the boundary may be obtained by superposition of a solution $u(x, y; t)$ which vanishes for t=0 and satisfies the prescribed boundary conditions, and a solution $v(x, y; t)$ which reduces to $f(x, y)$ for t=0 and satisfies the homogeneous boundary conditions obtained by replacing the second members of the above equations by 0. Moreover, a solution $u(x, y; t)$ which satisfies three nonhomogeneous boundary conditions of the above type for the three boundaries y=0, y=a and x=0 may evidently be obtained by superposition of three solutions, each one of which satisfies one nonhomogeneous and two homogeneous boundary conditions. For this reason, we shall confine ourselves to a number of typical problems involving one nonhomogeneous and two homogeneous boundary conditions; we will not, however, attempt to exhaust all possible combinations of boundary conditions of this type.

Case 1.

Boundary y=0 kept at temperature $\varphi(x; t)$; boundaries y=a and x=0 kept at 0°C. Initial temperature $f(x, y)$.

Examination of (15) shows that $u^*(x, y; p)$ the Laplace transform of the desired solution $u(x, y; t)$ may obtained by replacing $\cos\alpha(x - \xi)$ in (15) by $\cos\alpha(x - \xi) - \cos\alpha(x + \xi)$. Proceeding as in Section 1, Part 1, we finally get

$$u(x,y;t) = \frac{\sqrt{\pi k}}{a^2} \sum_{n=1}^{\infty} n \sin \frac{n\pi y}{a}$$

$$\cdot \int_0^{\infty} d\xi \int_0^{t} \left\{ e^{-\frac{(x-\xi)^2}{4k\tau}} - \delta\, e^{-\frac{(x+\xi)^2}{4k\tau}} \right\} \varphi(\xi; t-\tau) \cdot e^{-\frac{k n^2 \pi^2 \tau}{a^2}} \cdot \tau^{-\frac{3}{2}}\, d\tau$$

where $\delta = 1$. In entirely similar manner, we obtain

$$v(x,y;t) = \frac{1}{a\sqrt{\pi k t}} \sum_{n=1}^{\infty} \sin \frac{n\pi y}{a} \cdot e^{-\frac{k n^2 \pi^2 t}{a^2}}$$

$$\cdot \int_0^{\infty} \left\{ e^{-\frac{(x-\xi)^2}{4kt}} - \delta\, e^{-\frac{(x+\xi)^2}{4kt}} \right\} d\xi \int_0^{a} f(\xi,\eta) \sin \frac{n\pi\eta}{a}\, d\eta$$

with $\delta = 1$. It is readily seen that if in the expressions of $u(x,y;t)$ and $v(x,y;t)$ we put $\delta = -1$, the resulting expressions are the solutions appropriate to the case where the boundary x=0 is impervious to heat.

Case 2.

Boundary x=0 kept at temperature $\varphi(y;t)$; bounoaries y=0 and y=a kept at 0°C. Initial temperature $f(x,y)$.

It is readily seen that

$$u^*(x,y;p) = \frac{2}{a} \sum_{n=1}^{\infty} e^{-\alpha_n x} \sin \beta_n y \int_0^{a} \varphi^*(\eta;p) \sin \beta_n \eta\, d\eta$$

where $\beta_n = \frac{n\pi}{a}$ and $\alpha_n = \sqrt{\frac{p}{k} + \beta_n^2}$. Last equation yields:

$$u(x,y;t) = \frac{2}{a} \sum_{n=1}^{\infty} \sin \beta_n y \int_0^{a} \sin \beta_n \eta\, d\eta \int_0^{t} \varphi(\eta; t-\tau)$$

$$\cdot\, \psi_n(x;\tau,\beta_n)\, d\tau$$

where

$$\int_0^\infty e^{-pt}\,\psi_m(x;t,\beta_m)\,dt = e^{-x\sqrt{\frac{p}{k}+\beta_m^2}} .$$

It is known that [4] formula (30)

$$\psi_m(x;\tau,\beta_m) = \frac{x}{2\sqrt{\pi k}}\, e^{-k\beta_m^2\tau} \cdot e^{-\frac{x^2}{4k\tau}}\,\tau^{-\frac{3}{2}} .$$

The solution $u(x,y;t)$ thus finally becomes

$$u(x,y;t) = \frac{x}{2\sqrt{\pi k}} \cdot \sum_{m=1}^{\infty} \sin\frac{m\pi y}{a} \int_0^a \sin\frac{m\pi\eta}{a}\,d\eta$$

$$\cdot \int_0^t \varphi(\eta;t-\tau)\, e^{-\frac{k m^2\pi^2\tau}{a^2}} \cdot e^{-\frac{x^2}{4k\tau}}\,\tau^{-\frac{3}{2}}\,d\tau .$$

The solution $v(x,y;t)$ is evidently identical with that of Case 1.

Case 3.

Radiation at x=0 into a medium at temperature $\varphi(y;t)$; boundaries y=0 and y=a at 0°C. Initial temperature $f(x,y)$.

The Laplace transform $u^*(x,y;p)$ is given by

$$u^*(x,y;p) = \frac{2h}{a} \cdot \sum_{m=1}^{\infty} \frac{e^{-\alpha_m x}}{\alpha_m + h} \sin\beta_m y \int_0^a \varphi^*(\eta;p)\sin\beta_m\eta\,d\eta$$

with $\beta_m = \frac{m\pi}{a}$ and $\alpha_m = \sqrt{\frac{p}{k}+\beta_m^2}$, whence

$$u(x,y;t) = \frac{2h}{a}\sum_{m=1}^{\infty}\sin\beta_m y\int_0^a \sin\beta_m\eta\,d\eta\int_0^t \varphi(\eta;t-\tau)\,\psi_m(x;\tau,\beta_m)\,d\tau$$

where

$$\int_0^\infty e^{-pt}\,\psi_m(x;t,\beta_m) = \frac{e^{-x\sqrt{\frac{p}{k}+\beta_m^2}}}{h+\sqrt{\frac{p}{k}+\beta_m^2}} .$$

The solution of the last integral equation is [4], formula (67)

$$\psi_m(x;t,\beta_m) = \frac{e^{-k\beta_m^2 t}}{2t\sqrt{\pi k t}} \int_0^\infty e^{-hp}(x+p)\cdot e^{-\frac{(x+p)^2}{4kt}}\,dp .$$

The expression for u(x,y,t) thus finally becomes

$$u(x,y;t) = \frac{h}{a\sqrt{\pi k}} \sum_{m=1}^{\infty} \sin\frac{m\pi y}{a} \int_0^a \sin\frac{m\pi\eta}{a}\,d\eta$$

$$\cdot\int_0^t \varphi(\eta;t-\tau)\cdot e^{-\frac{k m^2\pi^2\tau}{a^2}}\,\tau^{-\frac{3}{2}}\,d\tau \int_0^\infty e^{-hp}\cdot e^{-\frac{(x+p)^2}{4k\tau}}(x+p)\,dp .$$

To obtain v(x,y,t), we make the substitution $v = v_1 + v_2$ where v_1 reduces to f(x,y) for t=0 and satisfies the condition $\frac{\partial v_1}{\partial x} = 0$ while v_2 vanishes for t=0 and satisfies the condition $\frac{\partial v_2}{\partial x} = h(v_1 + v_2)$. It is then clear that v_1 is identical with the solution v under Case 1, with $\delta = -1$. Also the expression for v_2 may be obtained from the above expression of u(x,y,t) by replacing $\varphi(y,t)$ by $-v_1(0,y,t)$ where $v_1(x,y,t)$ is the solution just considered.

Case 4.

Radiation at y=0 into a medium at temperature $\varphi(x;t)$; radiation at y=a into a medium at 0°C. Boundary x=0 kept at 0°C or impervious to heat. Initial temperature f(x,y).

The expression of $u^*(x,y;p)$ may be obtained from that of $u_1^*(x,y;p)$ of Section 5, Part I, by replacing $\cos\alpha(x-\xi)$ by $\cos\alpha(x-\xi) - \cos\alpha(x+\xi)$ in the case where the boundary x=0 is kept at 0°C and by $\cos\alpha(x-\xi) + \cos\alpha(x+\xi)$ in the case where the boundary x=0 is impervious to heat. This leads ultimately to

$$u(x,y;t) = \frac{h\sqrt{k}}{a\sqrt{\pi}} \sum_{m=1}^{\infty} \frac{\zeta_m^2\cos\left(1-\frac{y}{a}\right)\zeta_m + ah\,\zeta_m\sin\left(1-\frac{y}{a}\right)\zeta_m}{2(1+ah)\zeta_m\sin\zeta_m - \left\{ah(2+ah)-\zeta_m^2\right\}\cos\zeta_m}$$

$$\cdot\int_0^\infty d\xi\int_0^t e^{-\frac{k\tau\zeta_m^2}{a^2}}\left\{e^{-\frac{(x-\xi)^2}{4k\tau}} + \delta\, e^{-\frac{(x+\xi)^2}{4k\tau}}\right\}\varphi_1(\xi;t-\tau)\,\tau^{-\frac{1}{2}}\,d\tau$$

where $\delta = -1$ or $\delta = 1$ depending on whether the boundary x=0 is kept at 0° or is impervious to heat.

The desired solution v(x,y,t) may be obtained from (82) by replacing $e^{-\frac{(x-\xi)^2}{4kt}}$ by $e^{-\frac{(x-\xi)^2}{4kt}} + \delta e^{-\frac{(x+\xi)^2}{4kt}}$ where $\delta = -1$ or $\delta = 1$ depending on whether the boundary x=0 is kept at 0°C or is impervious to heat.

Case 5.

Boundary x=0 kept at temperature $\varphi(y;t)$; radiation at y=0 and y=a into a medium at 0°C. Initial temperature f(x,y).

Let $u = u_1 + u_2 + u_3$ where

$$u_3(0, y; t) = \varphi(y;t)$$

$$\frac{\partial u_3}{\partial y} = 0 \quad \text{for } y = 0 \text{ and } y = a$$

$$u_1(0, y; t) = u_2(0, y; t) = 0$$

$$\frac{\partial u_1}{\partial y} - h_1 u_1 = \begin{cases} h_1 u_3(x, 0; t) & \text{for } y = 0 \\ 0 & \text{for } y = a \end{cases}$$

$$\frac{\partial u_2}{\partial y} - h_2 u_2 = \begin{cases} h_2 u_3(x, a; t) & \text{for } y = a \\ 0 & \text{for } y = 0 \end{cases} .$$

The solution $u_3(x, y; t)$ may evidently be obtained from the expression for u(x,y;t) in Case 2 by replacing $\sin \frac{n\pi y}{a}$ by $\cos \frac{n\pi y}{a}$.

Thus

$$u_3(x, y; t) = \frac{x}{2\sqrt{\pi k}} \sum_{n=1}^{\infty} \cos \frac{n\pi y}{a} \int_0^a \cos \frac{n\pi\eta}{a} \, d\eta$$

$$\int_0^t \varphi(\eta; t-\tau) \cdot e^{-\frac{k n^2 \pi^2 \tau}{a^2}} e^{-\frac{x^2}{4k\tau}} \tau^{-\frac{3}{2}} \, d\tau .$$

The expression for $u_1(x, y; t)$ may be obtained from (86) by replacing $\varphi_1(x; t)$ by $-u_1(x, 0; t)$; similarly $u_2(x, y; t)$ may be obtained from (88) by replacing $\varphi_2(x; t)$ by $-u_1(x, a; t)$. Thus

$$u_1(x, y; t) = \frac{h_1\sqrt{\kappa}}{a\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{\zeta_n \{ a h_2 \sin (1-\frac{y}{a})\zeta_n - \zeta_n \cos (1-\frac{y}{a})\zeta_n \}}{\{1 + a(h_1 - h_2)\}\zeta_n \sin \zeta_n + \{a^2 h_1 h_2 + a(h_2 - h_1) + \zeta_n^2\} \cos \zeta_n}$$

$$\cdot \int_0^{\infty} d\xi \int_0^t e^{-\frac{\kappa\tau\zeta_n^2}{a^2}} \left\{ e^{-\frac{(x-\xi)^2}{4\kappa\tau}} + \delta e^{-\frac{(x+\xi)^2}{4\kappa\tau}} \right\} u_1(\xi, 0; t-\tau) \cdot \tau^{-\frac{1}{2}} \, d\tau ,$$

$$u_2(x, y; t) = \frac{h_2\sqrt{\kappa}}{a\sqrt{\pi}} \cdot \sum_{n=1}^{\infty} \frac{\zeta_n \{ a h_1 \sin \frac{y}{a}\zeta_n + \zeta_n \cos \frac{y}{a}\zeta_n \}}{\{1 + a(h_1 - h_2)\}\zeta_n \sin \zeta_n + \{a^2 h_1 h_2 + a(h_2 - h_1) + \zeta_n^2\} \cos \zeta_n}$$

$$\cdot \int_0^{\infty} d\xi \int_0^t e^{-\frac{\kappa\tau\zeta_n^2}{a^2}} \left\{ e^{-\frac{(x-\xi)^2}{4\kappa\tau}} + \delta e^{-\frac{(x+\xi)^2}{4\kappa\tau}} \right\} u_1(\xi, a; t-\tau) \cdot \tau^{-\frac{1}{2}} \, d\tau$$

where $\delta = -1$ and the summation extends over the roots of

$$(a^2 h_1 h_2 + \zeta^2) \tan \zeta - a(h_1 - h_2)\zeta = 0 .$$

By putting $h_2 = 0$ or $h_2 = \infty$ we obtain the solutions appropriate to the case where the boundary y=a is either impervious to heat, or kept at 0°C; similarly by putting $h_1 = 0$ or $h_1 = \infty$ we obtain the solutions appropriate to the case where the boundary y=0 is either impervious to heat or kept at 0°C. In the case where the radiation takes place at both boundaries y=0 and y=a into a medium at 0°C it is necessary to put $h_2 = -h_1$, for reasons previously explained.

To derive the solution v(x,y;t) we put $v = v_1 + v_2$ where

$$\lim_{t \to 0} v_1(x, y; t) = f(x, y)$$

$$\frac{\partial v_1}{\partial y} = 0 \quad \text{for} \quad y = 0 \text{ and } y = a$$

$$v_1 = 0 \quad \text{for} \quad x = 0$$

$$\lim_{t\to 0} v_2(x,y;t) = 0$$

$$\frac{\partial v_2}{\partial y} - h_1 v_2 = h_1 v_1(x,0;t) \quad \text{for } y = 0$$

$$\frac{\partial v_2}{\partial y} - h_2 v_2 = h_2 v_1(x,a;t) \quad \text{for } y = a$$

$$v_2 = 0 \qquad \text{for } x = 0 \; .$$

The expression for $v_2(x,y;t)$ may be obtained from (82) by replacing $e^{-\frac{(x-\xi)^2}{4kt}}$ by $e^{-\frac{(x-\xi)^2}{4kt}} - e^{-\frac{(x+\xi)^2}{4kt}}$. Thus

$$v_2(x,y;t) = \frac{1}{2a\sqrt{\pi k t}} \int_0^\infty \left\{ e^{-\frac{(x-\xi)^2}{4kt}} + \delta e^{-\frac{(x+\xi)^2}{4kt}} \right\} d\xi \int_0^a f(\xi,\eta)\, d\eta$$

$$+ \frac{1}{a\sqrt{\pi k t}} \sum_{n=1}^{\infty} e^{-\frac{k n^2 \pi^2 t}{a^2}} \cos\frac{n\pi y}{a} \int_0^\infty \left\{ e^{-\frac{(x-\xi)^2}{4kt}} + \delta e^{-\frac{(x+\xi)^2}{4kt}} \right\} d\xi$$

$$\cdot \int_0^a f(\xi,\eta) \cos\frac{n\pi\eta}{a}\, d\eta$$

with $\delta = -1$.

Comparison of the boundary conditions satisfied by $v_2(x,y;t)$ with those satisfied by u_1 and u_2 leads to the conclusion that

$$v_2(x,y;t) = \bar{u}_1(x,y;t) + \bar{u}_2(x,y;t)$$

where $\bar{u}_1$ and $\bar{u}_2$ are obtained from u_1 and u_2 by replacing $u_3(x,0;t)$ and $u_3(x,a;t)$ by $v_1(x,0;t)$ and $v_1(x,a;t)$ respectively. The remark made in connection with the solution of u_1 and u_2 above, applies of course also to the solution $v_2(x,y;t)$

Case C.

Temperature gradient $\varphi(y;t)$ on $x=0$; radiation at $y=0$ and $y=a$ into a medium at 0°C. Initial temperature $f(x,y)$.

In this case we put $u=u_1+u_2+u_3$ where

$$\frac{\partial}{\partial x} u_3(x, y; t) = \varphi(y; t) \qquad \text{for } x = 0$$

$$\frac{\partial}{\partial y} u_3(x, y; t) = 0 \qquad \text{for } y = 0 \text{ and } y = a$$

$$\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = 0 \qquad \text{for } x = 0$$

$$\frac{\partial u_1}{\partial y} - h_1 u_1 = \begin{cases} h_1 u_3(x, 0; t) & \text{for } y = 0 \\ 0 & \text{for } y = a \end{cases}$$

$$\frac{\partial u_2}{\partial y} - h_2 u_2 = \begin{cases} 0 & \text{for } y = 0 \\ h_2 u_3(x, a; t) & \text{for } y = a \end{cases} .$$

The Laplace transform of $u_3(x, y; t)$ is

$$u_3^*(x, y; p) = -\frac{2}{a} \sum_{m=1}^{\infty} \frac{e^{-\alpha_m x}}{\alpha_m} \cos \beta_m y \int_0^a \varphi^*(\eta, p) \cos \beta_m \eta \, d\eta$$

where $\beta_m = \frac{m\pi}{a}$ and $\alpha_m = \sqrt{\frac{p}{k} + \beta_m}$. The last equation yields

$$u_3(x, y; t) = -\frac{2}{a} \sum_{m=1}^{\infty} \cos \beta_m y \int_0^a \cos \beta_m \eta \, d\eta \, .$$

$$\cdot \int_0^t \varphi(\eta; t-\tau) \psi_m(x; \tau, \beta_m) \, d\tau$$

where

$$\int_0^{\infty} e^{-pt} \psi_m(x; t, \beta_m) = \frac{e^{-x\sqrt{\frac{p}{k}+\beta_m^2}}}{\sqrt{\frac{p}{k}+\beta_m^2}} .$$

It is known that the solution of the last integral equation is [4, p.12, equat.56

$$\psi(x; t, \beta_m) = \frac{\sqrt{k}}{\sqrt{\pi t}} e^{-k\beta_m^2 t} \cdot e^{-\frac{x^2}{4kt}} .$$

Thus the expression of $u_1(x, y; t)$ becomes

$$u_1(x, y; t) = -\frac{2\sqrt{k}}{a\sqrt{\pi}} \sum_{n=1}^{\infty} \cos\frac{n\pi x}{a} \int_0^a \cos\frac{n\pi\eta}{a}\, d\eta$$

$$\cdot \int_0^t \varphi(\eta; t-\tau) \cdot e^{-\frac{k n^2\pi^2\tau}{a^2}} \cdot e^{-\frac{y^2}{4k\tau}}\, \tau^{-\frac{1}{2}}\, d\tau .$$

Comparison of the boundary conditions satisfied by u_1 and u_2 with those satisfied by u_1 and u_2 in Case 5, leads to the conclusion that the desired solutions u_1 and u_2 are given by the expressions given under Case 5 with $\delta = 1$.

The expressions of u_1 and u_2 are identical with those of Case 5 except that now $\delta = 1$. The derivation of the solution $v(x,y,t)$ proceeds in exactly the same manner as in Case 5. Specifically, if we put $v = v_1 + v_2$ then v_1 is obtained from the corresponding expression under Case 5 by putting $\delta = 1$ and $v_2(x, y; t) = \bar{u}_1(x, y; t) + \bar{u}_2(x, y; t)$ where $\bar{u}_1$ and $\bar{u}_2$ are given by expressions identical with those of Case 5 except that $\delta = 1$. Finally the remarks made in connection with the solutions u_1, u_2 and v_2 of Case 5 (putting $h_2 = 0$, $h_2 = \infty$ or $h_2 = -h_1$) apply to the same solutions in the present case.

Part III. Heat Conduction in the Domain D_3'

The problems to be discussed below are the three-dimensional extensions of the problems discussed in Part I, to which we shall frequently have occasion to refer. As in Part II, we shall confine ourselves to the derivation of solutions of problems involving one nonhomogeneous and one homogeneous boundary condition. The meaning of the terms "U" solution and "V" solution are the same as in Part II.

Case 1.

Boundary $z=0$ kept at temperature $\varphi(x, y; t)$; boundary $z=a$ kept at 0°C. Initial temperature $f(x,y,z)$.

Derivation of solution u(x,y,z)

In view of the identity

$$\Phi(x, y) = \frac{1}{\pi^2} \iint_{-\infty}^{\infty} \Phi(\xi, \eta)\, d\xi\, d\eta \iint_0^{\infty} \cos\alpha(x-\xi) \cos\beta(y-\eta)\, d\alpha\, d\beta$$

it is readily seen that the expression

$$u^*(x,y,z;p) = \frac{1}{\pi^2}\iint_{-\infty}^{\infty} \varphi^*(\xi,\eta,p)\, d\xi\, d\eta \iint_0^{\infty} \frac{\sinh \gamma(a-z)}{\sinh \gamma a}$$

$$\cdot \cos\alpha(x-\xi)\cos\beta(y-\eta)\, d\alpha\, d\beta$$

where $\gamma = \sqrt{\frac{p}{k} + \alpha^2 + \beta^2}$ is the Laplace transform of the solution $u(x,y,z;t)$ vanishing for $t=0$ and satisfying the prescribed boundary conditions. The last equation yields

$$u(x,y,z;t) = \frac{1}{\pi^2}\iint_{-\infty}^{\infty} d\xi\, d\eta \int_0^t d\tau \iint_0^{\infty} \varphi(\xi,\eta;\, t-\tau)$$

$$\cdot \psi(z;\tau,\alpha,\beta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\, d\alpha\, d\beta$$

where

$$\int_0^{\infty} e^{-pt}\, \psi(z;t,\alpha,\beta)\, dt = \frac{\sinh \gamma(a-z)}{\sinh \gamma a}.$$

with

$$\gamma = \sqrt{\frac{p}{k} + \alpha^2 + \beta^2}\,.$$

By analogy with the developments in Section 1, Part I, we can write at once

$$\psi(z;t,\alpha,\beta) = e^{-k(\alpha^2+\beta^2)t} \cdot \Phi(z;t)$$

where

$$\Phi(z;t) = \frac{2\pi k}{a^2}\sum_{n=1}^{\infty} n \sin\frac{n\pi z}{a} \cdot e^{-\frac{k n^2\pi^2 t}{a^2}}\,.$$

With the aid of this expression for $\psi(z;\tau,\alpha,\beta)$ and of identity (20) the expression for u(x,y,z;t) finally becomes

$$u(x,y,z;t) = \frac{1}{2a^2}\sum_{n=1}^{\infty} n \sin\frac{n\pi z}{a} \iint_{-\infty}^{\infty} d\xi\, d\eta$$

$$\cdot \int_0^t \varphi(\xi,\eta;t-\tau)\cdot e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4k\tau}} \cdot e^{-\frac{k n^2\pi^2\tau}{a^2}} \cdot \tau^{-1}\, d\tau \;.$$

In view of the identity

$$\Phi(x,y,z) = \frac{1}{\pi^3}\iiint_{-\infty}^{\infty} d\xi\, d\eta\, d\zeta \iiint_0^{\infty} \Phi(\xi,\eta,\zeta)$$

$$\cdot \cos\alpha(x-\xi)\cos\beta(y-\eta)\cos\gamma(z-\zeta)\, d\alpha\, d\beta\, d\gamma$$

it follows that the expression

$$v^*(x,y,z;p) = \frac{1}{\pi^3}\iiint_{-\infty}^{\infty} d\xi\, d\eta\, d\zeta \iiint_0^{\infty} \Phi(\xi,\eta,\zeta)$$

$$\cdot \frac{\cos\alpha(x-\xi)\cos\beta(y-\eta)\cos\gamma(z-\zeta)}{k(\alpha^2+\beta^2+\gamma^2)+p}\, d\alpha\, d\beta\, d\gamma$$

where, by analogy with the developments in Section 1 Part I

$$\Phi(x,y,z+2ma) = f(x,y,z) \qquad 0 \le z \le a$$

$$\Phi(x,y,-z+2ma) = -f(x,y,z) \qquad m = \pm 1, \pm 2, \pm 3, \ldots \pm\infty$$

is the Laplace transform of the solution v(x,y,z;t) which vanishes for z=0 and z=a and reduces to f(x,y,z) for t=0. From the expression for v^* we get

$$v(x,y,z;t) = \frac{1}{\pi^3}\iiint_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta \iiint_0^{\infty} e^{-k(\alpha^2+\beta^2+\gamma^2)t}$$

$$\cdot \cos\alpha(x-\xi)\cos\beta(y-\eta)\cos\gamma(z-\zeta)\, d\alpha\, d\beta\, d\gamma \;.$$

With the aid of the identity (20) and the above relations satisfied by $\phi(x,y,z)$, the expression for $v(x,y,z;t)$ becomes

$$v(x,y,z;t) = \frac{1}{8(\pi k t)^{3/2}} \iint_{-\infty}^{\infty} f(\xi,\eta,\zeta)\, e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4kt}}\, d\xi\, d\eta$$

$$\cdot \int_0^a \left\{ \sum_{m=-\infty}^{\infty} \left[e^{-\frac{(z-\zeta-2ma)^2}{4kt}} - e^{-\frac{(z+\zeta-2ma)^2}{4kt}} \right] \right\} d\zeta \, .$$

In view of the identities (36) and (36') the last equation finally becomes

$$v(x,y,z;t) = \frac{1}{2\pi k t a} \sum_{m=1}^{\infty} e^{-\frac{k m^2 \pi^2 t}{a^2}} \sin\frac{m\pi z}{a}$$

$$\cdot \iint_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4kt}}\, d\xi\, d\eta \int_0^a f(\xi,\eta,\zeta) \sin\frac{m\pi\zeta}{a}\, d\zeta \, .$$

As a test of the correctness of the last result, it may be noted that if $f(x,y,z)$ becomes a function of z only, the last expression reduces, in view of (26) to

$$v(z;t) = \frac{2}{a} \sum_{m=1}^{\infty} \sin\frac{m\pi z}{a} \cdot e^{-\frac{k m^2 \pi^2 t}{a^2}} \int_0^a f(\zeta) \sin\frac{m\pi\zeta}{a}\, d\zeta$$

in agreement with the result given by Carslaw [1], p. 1̇̇0 .

Case 2.

Boundary $z=0$ kept at 0°C; temperature gradient $\varphi(x,y;t)$ prescribed on $z=a$. Initial temperature $f(x,y,z)$.

<u>Derivation of solution $u(x,y,z;t)$</u>

The counterpart of the expression u^* in Case 1 is

$$u^*(x,y,z,p) = \frac{1}{\pi^2} \iint_0^{\infty} d\alpha\, d\beta \iint_{-\infty}^{\infty} \varphi^*(\xi,\eta;p)\, \frac{\sinh \gamma z}{\gamma \cosh \gamma a}$$

$$\cdot \cos\alpha(x-\xi) \cos\beta(y-\eta)\, d\xi\, d\eta$$

where $\gamma = \sqrt{\frac{p}{k} + \alpha^2 + \beta^2}$. The last equation yields

$$u(x,y,z;t) = \frac{1}{\pi^2}\iint_{-\infty}^{\infty} d\xi\, d\eta \int_0^t d\tau \iint_{\tau}^{\infty} \varphi(\xi,\eta;t-\tau)$$

$$\cdot\, \psi(z;\tau,\alpha,\beta)\cos\alpha(x-\xi)\cos\beta(y-\eta)\, d\alpha\, d\beta$$

where

$$\int_0^{\infty} e^{-pt}\,\psi(z;t,\alpha,\beta)\,dt = \frac{\sinh \gamma z}{\gamma\cosh\gamma a}$$

with $\gamma = \sqrt{\frac{p}{k} + \alpha^2 + \beta^2}$. By analogy with the developments in Section 2 Part I we have

$$\psi(z;t,\alpha,\beta) = \frac{2k}{a}\, e^{-k(\alpha^2+\beta^2)t} \sum_{m=0}^{\infty} (-1)^m\, e^{-\frac{k(2m+1)^2\pi^2 t}{4a^2}} \cdot \sin\frac{(2m+1)\pi z}{2a} .$$

With the aid of this expression for $\psi(z;t,\alpha,\beta)$ and of the identity (20), the expression for $u(x,y,z;t)$ becomes

$$u(x,y,z;t) = \frac{1}{2\pi a}\sum_{m=0}^{\infty}(-1)^m \sin\frac{(2m+1)\pi z}{2a}\int_0^t \frac{d\tau}{\tau}$$

$$\iint_{-\infty}^{\infty}\varphi(\xi,\eta;t-\tau)\cdot e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4k\tau}}\cdot e^{-\frac{k(2m+1)^2\pi^2\tau}{4a^2}}\, d\xi\, d\eta .$$

<u>Derivation of solution $v(x,y,z;t)$</u>

As in Section 2 Part I the solution $v(x,y,z;t)$ is identical with the solution of the problem of heat conduction in a slab of thickness 2a, whose bounding planes $z=0$ and $z=2a$ are kept at 0°C, initially at a temperature $\phi(x,y,z)$ defined by

$$\phi(x,y,z) = f(x,y,z) \qquad \text{for} \qquad 0 < z < a \quad \text{and}$$

$$\phi(x,y,2a-z) = f(x,y,z) \qquad \text{for} \qquad 0 < z < 2a .$$

Starting with the solution $v(x,y,z;t)$ in Case 1 we ultimately get

$$v(x,y,z;t) = \frac{1}{2\pi k t a} \sum_{n=0}^{\infty} \sin \frac{(2n+1)\pi z}{2a} \cdot e^{-\frac{k(2n+1)^2\pi^2 t}{4a^2}}$$

$$\cdot \iint_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4kt}} d\xi \, d\eta \int_0^a f(\xi,\eta,\zeta) \sin \frac{(2n+1)\pi\zeta}{2a} d\zeta .$$

Case 3.

Radiation at the boundary z=0 into a medium at temperature $\varphi(x,y;t)$; boundary z=a kept at 0°C. Initial temperature $f(x,y,z)$.

The counterpart of solution u_1^* in Section 3 Part I is

$$u^*(x,y,z;p) = \frac{h}{\pi^2} \iint_0^{\infty} d\alpha \, d\beta \iint_{-\infty}^{\infty} \varphi^*(\xi,\eta;p) \cdot \frac{\sinh \gamma(z-a)}{\gamma \cosh \gamma a + h \sinh \gamma a}$$

$$\cos \alpha(x-\xi) \cos \beta(y-\eta) \, d\xi \, d\eta$$

where $\gamma = \sqrt{\frac{p}{k} + \alpha^2 + \beta^2}$. The last equation yields

$$u(x,y,z;t) = \frac{h}{\pi^2} \iint_{-\infty}^{\infty} d\xi \, d\eta \int_0^t d\tau \iint_0^{\infty} \varphi(\xi,\eta;t-\tau)$$

$$\cdot \psi(z;\tau,\alpha,\beta) \cos \alpha(x-\xi) \cos \beta(y-\eta) \, d\alpha \, d\beta$$

where now

$$\int_0^{\infty} e^{-pt} \psi(z;t,\alpha,\beta) \, dt = \frac{\sinh \gamma(z-a)}{\gamma \cosh \gamma a + h \sinh \gamma a} .$$

By analogy with the developments in Section 3 Part I we have

$$\psi(z; t, \alpha, \beta) = 2hk\, e^{-k(\alpha^2+\beta^2)t} \sum_{m=1}^{\infty} \frac{\zeta_m \sin(1-\frac{z}{a})\zeta_m\, e^{-\frac{kt\zeta_m^2}{a^2}}}{\{(1+ah)ah + \zeta_m^2\}\cos\zeta_m}$$

where the summation extends over the roots of the transcendental equation

$$\zeta + ah \tan\zeta = 0 .$$

Substituting the above expression of $\psi(z; t, \alpha, \beta)$ in the expression for $u(x,y,z;t)$ and making use of the identity (20), we ultimately get

$$u(x,y,z;t) = \frac{h}{2\pi} \cdot \sum_{m=1}^{\infty} \frac{\zeta_m \sin(1-\frac{z}{a})\zeta_m}{\{(1+ah)ah + \zeta_m^2\}\cos\zeta_m}$$

$$\int_0^t e^{-\frac{k\tau\zeta_m^2}{a^2}} \frac{d\tau}{\tau} \iint_{-\infty}^{\infty} \varphi(\xi,\eta; t-\tau) \cdot e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4k\tau}}\, d\xi\, d\eta$$

where the summation extends over the roots of $\zeta + ah\tan\zeta = 0$.

<u>Derivation of solution $v(x,y,z;t)$</u>

By analogy with the developments in Section 3 Part I, we put

$$v(x,y,z;t) = v_1(x,y,z;t) + v_2(x,y,z;t)$$

where

$$\lim_{t\to 0} v_1(x,y,z;t) = f(x,y,z)$$

$$v_1(x,y,a,t) = 0$$

$$\frac{\partial}{\partial z} v_1(x,y,z;t) = 0 \qquad \text{for} \quad z = 0$$

$$\lim_{t\to 0} v_2(x,y,z;t) = 0$$

$$v_2(x,y,a;t) = 0$$

$$\left(\frac{\partial}{\partial z} - h\right) v_2(x,y,z;t) = h v_1(x,y,0;t) \qquad \text{for } z = 0 .$$

Derivation of solution $v_1(x,y,z;t)$

By analogy with the developments in Section 3 Part I if we define:

$$\left\{\begin{aligned} \Phi(x,y,\pm z+4ma) &= f(x,y,z) \\ \Phi(x,y,\pm z+4ma+a) &= -f(x,y,z) \end{aligned}\right\} \qquad \begin{aligned} &0 < z < a \\ &m = \pm 1, \pm 2, \pm 3, \ldots \pm \infty \; . \end{aligned}$$

then the solution $v(x,y,z;t)$ becomes successively

$$v(x,y,z;t) = \frac{1}{\pi^3}\iiint\limits_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta \iiint\limits_{0}^{\infty} e^{-k(\alpha^2+\beta^2+\gamma^2)t}$$

$$\cdot \cos\alpha(x-\xi)\cos\beta(y-\eta)\cos\gamma(z-\zeta)\; d\alpha\, d\beta\, d\gamma$$

$$= \frac{1}{8(\pi k t)^{3/2}} \iiint\limits_{-\infty}^{\infty} \Phi(\xi,\eta,\zeta)\; e^{-\frac{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}{4kt}}\; d\xi\, d\eta\, d\zeta$$

$$= \frac{1}{8(\pi k t)^{3/2}} \iint\limits_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4kt}}\; d\xi\, d\eta$$

$$\cdot \int_0^a f(\xi,\eta,\zeta) \left\{ \sum_{m=-\infty}^{\infty}\left[e^{-\frac{(z+\zeta-4ma)^2}{4kt}} + e^{-\frac{(z-\zeta-4ma)^2}{4kt}} \right]\right.$$

$$\left. - \sum_{-\infty}^{\infty}\left[e^{-\frac{(z+\zeta-4ma-2a)^2}{4kt}} + e^{-\frac{(z-\zeta-4ma-2a)^2}{4kt}} \right]\right\} d\zeta \; .$$

Making use of the identities (36) and (36') the last equation ultimately becomes

$$v(x,y,z;t) = \frac{1}{2\pi a k t} \cdot \sum_{m=0}^{\infty} e^{-\frac{(2m+1)^2\pi^2 k t}{4a^2}} \cos\frac{(2m+1)\pi z}{2a}$$

$$\cdot \iint\limits_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4kt}}\; d\xi\, d\eta \int_0^a f(\xi,\eta,\zeta)\cos\frac{(2m+1)\pi\zeta}{2a}\; d\zeta \; .$$

Derivation of solution $v_2(x,y,z;t)$

It is clear that the expression for $v_2(x,y,z;t)$ may be obtained from the expression for $u(x,y,z;t)$ by replacing $\varphi(x,y;t)$ by $-v_1(x,y,0;t)$.

Case 4.

Radiation at the boundary $z=0$ into a medium at temperature $\varphi(x,y;t)$; boundary $z=a$ impervious to heat. Initial temperature $f(x,y,z)$.

Derivation of solution $u(x,y,z;t)$

The formal expression for $u(x,y,z;t)$ is identical with that in the previous cases, except that $\psi(z;t,\alpha,\beta)$ must be obtained from

$$\int_0^\infty e^{-pt}\,\psi(z;t,\alpha,\beta)\,dt = \frac{\cosh\gamma(z-a)}{\gamma\cosh\gamma a + h\sinh\gamma a}\,.$$

Proceeding as in Section 4 Part I we get

$$\psi(z;t,\alpha,\beta) = 2hk\,e^{-k(\alpha^2+\beta^2)t}\cdot\sum_{n=1}^{\infty}\frac{\zeta_n e^{-\frac{kt\zeta_n^2}{a^2}}\cos\left(1-\frac{z}{a}\right)\zeta_n}{\{ah(1+ah)+\zeta_n^2\}\sin\zeta_n}$$

where the summation extends over the roots of

$$\zeta\tan\zeta = ah\,.$$

With the above expression for $\psi(z;t,\alpha,\beta)$ we ultimately get

$$u(x,y,z;t) = \frac{h^2}{2\pi}\cdot\sum_{n=1}^{\infty}\frac{\zeta_n\cos\left(1-\frac{z}{a}\right)\zeta_n}{\{ah(1+ah)+\zeta_n^2\}\cos\zeta_n}$$

$$\cdot\int_0^t e^{-\frac{k\tau\zeta_n^2}{a^2}}\cdot e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4k\tau}}\,\frac{d\tau}{\tau}\iint_{-\infty}^{\infty}\varphi(\xi,\eta;t-\tau)\,d\xi\,d\eta$$

where the summation extends over the roots of the above transcendental equations.

Derivation of solution $v(x,y,z;t)$

As in the previous case we put

$$v(x,y,z;t) = v_1(x,y,z;t)+v_2(x,y,z;t)$$

where

$$\lim_{t\to 0} v_1(x,y,z;t) = f(x,y,z)$$

$$\frac{\partial}{\partial z} v_1(x,y,z;t) = 0 \qquad \text{for} \quad z = 0 \text{ and } z = a$$

$$\lim_{t\to 0} v_2(x,y,z;t) = 0$$

$$\left(\frac{\partial}{\partial z} - h\right) v_2(x,y,z;t) = h\, v_1(x,y,0;t) \qquad \text{for } z = 0$$

$$\frac{\partial}{\partial z} v_2(x,y,z;t) = 0 \qquad \text{for} \quad z = a \,.$$

Derivation of solution $v_1(x,y,z,t)$

The derivation of $v_1(x,y,z;t)$ follows very closely the developments in Section 4 Part I and those of the preceding case. If we define

$$\left\{\begin{array}{l} \Phi(x,y,z+2ma) = f(x,y,z) \\ \Phi(x,y,z+2ma+a) = f(x,y,a-z) \end{array}\right\} \qquad \begin{array}{l} 0 \le z \le a \\ m = 0, \pm 1, \pm 2, \dots \pm\infty \end{array}$$

then

$$v_1(x,y,z;t) = \frac{1}{8(\pi k t)^{\frac{3}{2}}} \iint_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4kt}}\, d\xi\, d\eta$$

$$\cdot \int_0^a f(\xi,\eta,\zeta) \left\{ \sum_{m=-\infty}^{\infty} \left[e^{-\frac{(z-\zeta-2ma)^2}{4kt}} + e^{-\frac{(z+\zeta-2ma)^2}{4kt}} \right]\right\} d\zeta \,.$$

With the aid of (36) and (36') the last equation becomes

$$v_1(x,y,z;t) = \frac{1}{4\pi a k t} \iint_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4kt}}\, d\xi\, d\eta \int_0^a f(\xi,\eta,\zeta)\, d\zeta$$

$$+ \frac{1}{2\pi a k t} \sum_{n=1}^{\infty} e^{-\frac{k n^2 \pi^2 t}{a^2}} \cos\frac{n\pi z}{a} \iint_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4kt}}\, d\xi\, d\eta$$

$$\cdot \int_0^a f(\xi,\eta,\zeta) \cos\frac{n\pi\zeta}{a}\, d\zeta \,.$$

Derivation of solution $v_2(x,y,z;t)$

It is clear that the expression for $v_2(x,y,z;t)$ may be obtained from the expression for $u(x,y,z;t)$ by replacing $\varphi(x,y;t)$ by $-v_1(x,y,0;t)$.

Case 5.

Boundary $z=0$ radiating into a medium at temperature $\varphi_0(x,y;t)$; boundary $z=a$ radiating into a medium at $0°C$. Initial temperature $0°C$.

As in the preceding four cases, the solution $u(x,y,z;t)$ is given by

$$u(x,y,z;t) = \frac{1}{\pi^2} \iint_{-\infty}^{\infty} d\xi \, d\eta \int_0^t d\tau \iint_0^{\infty} \varphi(\xi,\eta;t-\tau)$$

$$\cdot \psi(z;\tau,\alpha,\beta) \cos\alpha(x-\xi) \cos\beta(y-\eta) \, d\alpha \, d\beta \, .$$

By analogy with the developments in Section 5 Part I the function $\psi(z;t,\alpha,\beta)$ is obtained from the inversion of

$$\int_0^{\infty} e^{-pt} \psi(z;t,\alpha,\beta) \, dt = \frac{h_1\{\gamma \cosh\gamma(z-a) + h_2 \sinh\gamma(z-a)\}}{(\gamma^2 - h_1 h_2)\sinh\gamma a + (h_1 - h_2)\gamma \cosh\gamma a}$$

where $\gamma = \sqrt{\frac{p}{k} + \alpha^2 + \beta^2}$. The inversion of last equation yields

$$\psi(z;t,\alpha,\beta) = \frac{2h_1 k}{a} \sum_{n=1}^{\infty} e^{-\frac{k t \zeta_n^2}{a^2}} \cdot e^{-k(\alpha^2+\beta^2)t}$$

$$\cdot \frac{\zeta_n\{a h_2 \sin(1-\frac{z}{a})\zeta_n - \zeta_n \cos(1-\frac{z}{a})\zeta_n\}}{\{2 + a(h_1 - h_2)\}\zeta_n \sin\zeta_n + \{a^2 h_1 h_2 + a(h_1 - h_2) + \zeta_n^2\}\cos\zeta_n}$$

where the ζ_n's are the roots of the transcendental equation

$$(a^2 h_1 h_2 + \zeta^2)\tan\zeta - a(h_1 - h_2)\zeta = 0 \, .$$

Substituting the above expresssion of ψ in the expression of u and making use of (20) we obtain:

$$u(x, y, z; t) = \frac{h_1}{2\pi a} \cdot \sum_{n=1}^{\infty} \frac{\zeta_n \{ a h_2 \sin (1 - \frac{z}{a}) \zeta_n - \zeta_n \cos (1 - \frac{z}{a}) \zeta_n \}}{\{ 2 + a(h_1 - h_2)\} \zeta_n \sin \zeta_n + \{ a^2 h_1 h_2 + a(h_2 - h_1) + \zeta_n^2 \} \cdot \cos \zeta_n}$$

$$\cdot \int_0^t e^{-\frac{k\tau\zeta_n^2}{a^2}} \cdot e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4k\tau}} \cdot \frac{d\tau}{\tau} \iint_{-\infty}^{\infty} \varphi(\xi, \eta; t-\tau) . d\xi \, d\eta$$

where the summation extends over the roots of the above transcendental equation.

If in the expression for u we put $h_1 = 0$ we obtain the solution appropriate to the case where the boundary $z=a$ is impervious to heat; in this case the summation must of course be extended over the roots of the transcendental equation obtained by putting $h_1 = 0$ in the above transcendental equation. Similarly if in the expression of u and in the transcendental equation we put $h_2 = \infty$, we obtain the solution appropriate to the case where the boundary $z=a$ is kept at 0°C. Finally it should be remarked that when the boundary $z=a$ actually radiates into a medium at 0°C we must put $h_2 = -h_1$ both in the expression for u as well as in the transcendental equation; the reason was explained at the end of Part I.

Case 6.

Boundary $z=a$ radiating into a medium at temperature $\varphi(x,y;t)$; boundary $z=0$ radiating into a medium at 0°C. Initial temperature 0°C.

The procedure is entirely similar to that of Case 5. The expression for u is in fact formally identical with that in formula (A) except that the function ψ, by analogy with the developments in Section 5 Part I is now obtained by the inversion of

$$\int_0^{\infty} e^{-pt} \psi(z; t, \alpha, \beta) \, dt = \frac{h_1 \{ \gamma \cosh \gamma z + h_1 \sinh \gamma z \}}{(\gamma^2 - h_1 h_2) \sinh \gamma a + (h_1 - h_2) \gamma \cosh \gamma a} .$$

This equation yields

$$\psi(z; t, \alpha, \beta) = \frac{2 h_1 k}{a} \cdot \sum_{n=1}^{\infty} e^{-\frac{kt\zeta_n^2}{a^2}} \cdot e^{-k(\alpha^2+\beta^2)t}$$

$$\frac{\zeta_n \{ \zeta_n \cos \frac{z}{a} \zeta_n + a h_1 \sin \frac{z}{a} \zeta_n \}}{\{ 2 + a(h_1 - h_2)\} \zeta_n \sin \zeta_n + \{ a^2 h_1 h_2 + a(h_2 - h_1) + \zeta_n^2 \} \cos \zeta_n}$$

whence ultimately

$$u(x, y, z; t) = \frac{h_2}{2\pi a} \sum_{m=1}^{\infty} \frac{\zeta_m \{ \zeta_m \cos \frac{z}{a} \zeta_m + a h_1 \sin \frac{z}{a} \zeta_m \}}{\{2 + a(h_1 - h_2)\} \zeta_m \sin \zeta_m + \{ a^2 h_1 h_2 + a(h_2 - h_1) + \zeta_m^2 \} \cos \zeta_m}$$

$$\cdot \int_0^t e^{-\frac{k\tau\zeta_m^2}{a^2}} \cdot e^{-\frac{(x-\xi)^2 + (y-\eta)^2}{4k\tau}} \cdot \frac{d\tau}{\tau} \iint_{-\infty}^{\infty} \varphi(\xi, \eta; t-\tau)\, d\xi\, d\eta ,$$

the summation in the last two equations being extended over the roots of the same transcendental equation as in the previous case. If we put $h_1 = 0$ or $h_1 = \infty$ we obtain the solutions appropriate to the cases where the boundary $z=0$ is either impervious to heat or kept at 0°C. Finally for reasons previously mentioned the factor h_1 in the above expressions of ψ and u must be replaced by $-h_1$.

Accordingly the desired solution actually becomes

$$u(x, y, z; t) = \frac{h}{2\pi a} \sum_{m=1}^{\infty} \frac{\zeta_m \{ \zeta_m \cos \frac{z}{a} \zeta_m + a h \sin \frac{z}{a} \zeta_m \}}{2(1 + ah) \zeta_m \sin \zeta_m - (a^2 h^2 + 2ah - \zeta_m^2) \cos \zeta_m}$$

$$\cdot \int_0^t e^{-\frac{k\tau\zeta_m^2}{a^2}} \cdot e^{-\frac{(x-\xi)^2 + (y-\eta)^2}{4k\tau}} \frac{d\tau}{\tau} \iint_{-\infty}^{\infty} \varphi(\xi, \eta; t-\tau)\, d\xi\, d\eta$$

where the summation extends over the roots of the transcendental equation

$$(\zeta^2 - a^2 h^2) \tan \zeta - 2ah\zeta = 0 .$$

(In the above two equations we have written h for h_1.)

Case 7.

Initial temperature f(x,y,z); boundaries $z=0$ and $z=a$ radiating into a medium at 0°C.

We put $v=v_1+v_2+v_3$ where

$$\lim_{t \to 0} v_1(x, y, z; t) = f(x, y, z)$$

$$\frac{dv_1}{dz} = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad z = a$$

$$\lim_{t \to 0} v_2(x, y, z; t) = \lim_{t \to 0} v_3(x, y, z; t) = 0$$

$$\frac{dv_2}{dz} = h_1 (v_2 + v_1) \qquad \text{for } z = 0$$

$$\frac{dv_2}{dz} = - h v_2 \qquad \text{for } z = a$$

$$\frac{dv_3}{dz} = h v_3 \qquad \text{for } z = 0$$

$$\frac{dv_3}{dz} = - h (v_3 + v_1) \qquad \text{for } z = a .$$

It is then readily seen that:

The function $v_1(x,y,z;t)$ is identical with that of Case 4.

The expression for $v_2(x,y,z;t)$ may be obtained from the expression of $u(x,y,z;t)$ of Case 5 by replacing $\varphi(x,y;t)$ by $-v_1(x,y,0;t)$.

The expression for $v_3(x,y,z;t)$ may be obtained from the expression for $u(x,y,z;t)$ of Case 6 by replacing $\varphi(x,y;t)$ by $-v_1(x,y,a;t)$. In obtaining the expressions for v_2 and v_3, it should be noted that $h_2 = -h_1$.

Part IV. Heat Conduction in the Domain D_3" $(-\infty < x < \infty,\ 0 < y < \infty,\ 0 < z < a)$

The problems to be discussed below are the three-dimensional extensions of those in Part II. Once more we shall confine ourselves to the derivation of solutions of problems involving one nonhomogeneous and (in this case) two homogeneous boundary conditions without however attempting to exhaust all possible combinations of boundary conditions of this type. As heretofore a "u" solution signifies a solution vanishing for $t=0$ and satisfying the prescribed boundary conditions; also a "v" solution denotes a solution satisfying the prescribed initial condition, i.e., in this case reducing to the function $f(x,y,z)$ for $t=0$, and three homogeneous boundary conditions, two of which are identical with those originally given and the third being obtained from the given nonhomogeneous boundary condition by replacing the second member of the equation expressing it by zero.

The subsequent developments will follow quite closely the developments in Part II.

It is readily seen that from the solutions of the problems in Part III we may derive at once the solutions of corresponding problems for the domain under consideration. Specifically by replacing $\int_{-\infty}^{\infty} d\eta$ by $\int_0^{\infty} d\eta$ and the factor $e^{-\frac{(y-\eta)^2}{4\kappa t}}$ by $e^{-\frac{(y-\eta)^2}{4\kappa t}} - e^{-\frac{(y+\eta)^2}{4\kappa t}}$ we obtain the solution of a problem in which it is required that the boundary $y=0$ be kept at $0°C$. Similarly by replacing $\int_{-\infty}^{\infty} d\eta$ by $\int_{0}^{\infty} d\eta$ and $e^{-\frac{(y-\eta)^2}{4Kt}}$ by $e^{-\frac{(y-\eta)^2}{4Kt}} + e^{-\frac{(y+\eta)^2}{4Kt}}$

we obtain the solution of a problem in which it is required that the boundary y=0 be impervious to heat. It will therefore suffice to discuss problems in which the boundary y=0 is neither kept at 0°C nor impervious to heat.

Case 1.

Boundary y=0 kept at temperature φ (x,z;t); boundaries z=0 and z=a kept at 0°C. Initial temperature f(x,y,z).

For a function ϕ (x,z) defined in the domains $-\infty < x < \infty$, $0 < z < a$ we have the representation

$$\phi(x,z) = \frac{2}{\pi a} \cdot \sum_{m=1}^{\infty} \sin \gamma_m z \int_0^{\infty} d\alpha \int_0^a \sin \gamma_m \zeta \, d\zeta \cdot \int_{-\infty}^{\infty} \varphi(\xi,\zeta) \cos \alpha (x-\xi) \, d\xi$$

where $\gamma_m = \frac{m\pi}{a}$. From this identity it follows that the Laplace transform of u(x,y,z;t) is

$$u^*(x,y,z;p) = \frac{2}{\pi a} \sum_{m=1}^{\infty} \sin \gamma_m z \int_0^{\infty} d\alpha \int_0^a \sin \gamma_m \zeta \, d\zeta \cdot \int_{-\infty}^{\infty} e^{-\beta_m y} \varphi^*(\xi,\zeta;p) \cos \alpha (x-\xi) \, d\xi$$

where $\beta_m = \sqrt{\frac{p}{k} + \alpha^2 + \gamma_m^2}$. The last equation yields

$$u(x,y,z;t) = \frac{2}{\pi a} \sum_{m=1}^{\infty} \sin \gamma_m z \int_0^{\infty} d\alpha \int_0^a \sin \gamma_m \zeta \, d\zeta \tag{A}$$

$$\cdot \int_{-\infty}^{\infty} \cos \alpha (x-\xi) \, d\xi \int_0^t \varphi(\xi,\zeta;t-\tau) \, \psi(y;\tau,\alpha,\gamma_m) \, d\tau$$

where

$$\int_0^{\infty} e^{-pt} \psi(y;t,\alpha,\gamma_m) \, dt = e^{-y\sqrt{\frac{p}{k} + \alpha^2 + \gamma_m^2}}.$$

It is known that

$$\psi(y;\tau,\alpha,\gamma_n) = \frac{y}{2\sqrt{\pi k}}\, e^{-\frac{y^2}{4k\tau}} \cdot e^{-k(\alpha^2+\gamma_n^2)\tau} \cdot \tau^{-3/2} .$$

(See equation (31) in 4)

Substituting the above expression of ψ in the expression for u and making use of (20) we ultimately get

$$u(x,y,z;t) = \frac{y}{2\pi a k} \sum_{n=1}^{\infty} \sin\frac{n\pi z}{a} \int_0^a \sin\frac{n\pi\zeta}{a}\, d\zeta \int_{-\infty}^{\infty} d\xi$$

$$\cdot \int_0^t e^{-\frac{y^2}{4k\tau}} \cdot e^{-\frac{k n^2\pi^2\tau}{a^2}}\, \varphi(\xi,\zeta;t-\tau) \cdot \tau^{-3/2}\, d\tau .$$

It is obvious that the solution [redacted](x,y,z;t) may be obtained from the corresponding expression in Part III Section 1 by replacing $\int_{-\infty}^{\infty} d\eta$ by $\int_0^{\infty} d\eta$ and the factor $e^{-\frac{(y-\eta)^2}{4kt}}$ by $e^{-\frac{(y-\eta)^2}{4kt}} - e^{-\frac{(y+\eta)^2}{4kt}}$. Thus

$$v(x,y,z;t) = \frac{1}{2\pi a k t} \sum_{n=1}^{\infty} \sin\frac{n\pi z}{a} \cdot e^{-\frac{k n^2\pi^2 t}{a^2}}$$

$$\cdot \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}}\, d\xi \int_0^{\infty} \left\{ e^{-\frac{(y-\eta)^2}{4kt}} - e^{-\frac{(y+\eta)^2}{4kt}} \right\} d\eta$$

$$\int_0^a f(\xi,\eta,\zeta)\, \sin\frac{n\pi\zeta}{a}\, d\zeta .$$

Case 2.

Radiation at the boundary y=0 into a medium at temperature φ(x,z,t); boundaries z=0 and z=a kept at 0°C. Initial temperature f(x,y,z).

Comparing the present problem with that discussed in Part II Case 3, we are led to the conclusion that the Laplace transform u* is given by

$$u^*(x,y,z;p) = \frac{2h}{\pi a} \sum_{n=1}^{\infty} \sin \gamma_n z \int_0^{\infty} d\alpha \int_0^a \sin \gamma_n \zeta\, d\zeta$$

$$\cdot \int_{-\infty}^{\infty} \frac{e^{-\beta_n y}}{\beta_n + h} \cdot \varphi^*(\xi,\zeta;p) \cos \alpha(x-\xi)\, d\xi .$$

It follows that $u(x,y,z;t)$ is given formally by equation (A) of the previous case, where now ψ is given by

$$\int_0^\infty e^{-pt}\, \psi(y;t,\alpha,\gamma_m)\, dt = \frac{h\, e^{-\beta_m y}}{\beta_m + h}$$

where

$$\beta_m = \sqrt{\frac{p}{k} + \alpha^2 + \gamma_m^2} .$$

Comparing with the developments in Part II Case 3, we conclude

$$\psi(y;t,\alpha,\gamma_m) = \frac{h\, e^{-k(\alpha^2+\gamma_m^2)t}}{2t\sqrt{\pi k t}} \int_0^\infty e^{-h\rho}\,(y+\rho)\, e^{-\frac{(y+\rho)^2}{4kt}}\, d\rho .$$

Substituting this expression for ψ in (A) and making use of (20) we obtain

$$u(x,y,z;t) = \frac{h}{2\pi a k} \cdot \sum_{m=1}^{\infty} \sin\frac{m\pi z}{a} \int_0^a \sin\frac{m\pi\zeta}{a}\, d\zeta \int_{-\infty}^{\infty} d\xi$$

$$\cdot \int_0^t e^{-\frac{(x-\xi)^2}{4k\tau}}\, e^{-\frac{k m^2\pi^2\tau}{a^2}}\, \varphi(\xi,\zeta;t-\tau)\,\tau^{-2}\, d\tau \int_0^\infty e^{-h\rho}\,(y+\rho)\, e^{-\frac{(y+\rho)^2}{4k\tau}}\, d\rho .$$

The method of obtaining the solution $v(x,y,z;t)$ is entirely similar with that of Part II Case 3. We have $v = v_1 + v_2$ where

$$v_1(x,y,z;t) = \frac{1}{2\pi a k t} \sum_{m=1}^{\infty} \sin\frac{m\pi z}{a}\, e^{-\frac{k m^2\pi^2 t}{a^2}}$$

$$\cdot \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}}\, d\xi \int_0^\infty \left\{ e^{-\frac{(y-\eta)^2}{4kt}} + e^{-\frac{(y+\eta)^2}{4kt}} \right\} d\eta$$

$$\cdot \int_0^a f(\xi,\eta,\zeta)\, \sin\frac{m\pi\zeta}{a}\, d\zeta$$

and where $v_2(x,y,z;t)$ may be obtained from the above expression of $u(x,y,z;t)$ by replacing $\varphi(x,z;t)$ by $-v_1(x,0,z;t)$.

Case 3.

Boundary $y=0$ kept at temperature $\varphi(x,z;t)$; radiation at the boundaries $z=0$ and $z=a$ into a medium at $0°C$. Initial temperature $f(x,y,z)$.

The method of deriving the solutions $u(x,y,z;t)$ and $v(x,y,z;t)$ is entirely similar to that of Case 5 Part II. For the derivation of the "u" solution we put $u = u_1 + u_2 + u_3$ where

$$u_3(x,0,z;t) = \varphi(x,z;t)$$

$$\frac{\partial u_3}{\partial z} = 0 \quad \text{for} \quad z=0 \text{ and } z=a$$

$$\frac{\partial u_1}{\partial z} - \lambda_1 u_1 = \begin{cases} \lambda u_3(x,y,0;t) & \text{for } z=0 \\ 0 & \text{for } z=a \end{cases}$$

$$\frac{\partial u_2}{\partial z} - \lambda_2 u_2 = \begin{cases} 0 & \text{for } z=0 \\ \lambda u_3(x,y,a;t) & \text{for } z=a \end{cases}$$

$$u_1(x,0,z;t) = u_2(x,0,z;t) = 0 .$$

The solution $u_3(x,y,z;t)$ may be obtained from the solution $u(x,y,z;t)$ corresponding to Case 1 above, by replacing $\sin\frac{n\pi z}{a}$ and $\sin\frac{n\pi\zeta}{a}$ by $\cos\frac{n\pi z}{a}$ and $\cos\frac{n\pi\zeta}{a}$ respectively. Thus:

$$u_3(x,y,z;t) = \frac{y}{2\pi a k}\sum_{n=1}^{\infty}\cos\frac{n\pi z}{a}\int_0^a \cos\frac{n\pi\zeta}{a}\,d\zeta\int_{-\infty}^{\infty}d\xi$$

$$\cdot\int_0^t e^{-\frac{y^2}{4k\tau}}\cdot e^{-\frac{kn^2\pi^2\tau}{a^2}}\,\varphi(\xi,\zeta;t-\tau)\,\tau^{-2}\,d\tau .$$

The solution $u_1(x,y,z;t)$ may be obtained from that corresponding to Case 5 Part III by replacing $\iint_{-\infty}^{\infty} d\xi\, d\eta$ by $\int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta$, $\varphi(x,y;t)$ by $-u_3(x,y,0;t)$ and $e^{-\frac{(y-\eta)^2}{4k\tau}}$ by $e^{-\frac{(y-\eta)^2}{4k\tau}} - e^{-\frac{(y+\eta)^2}{4k\tau}}$. Thus:

$$u_1(x,y,z;t) = \frac{\lambda_1}{2\pi a}\sum_{n=1}^{\infty}\frac{\zeta_n\{a\lambda_2\sin(1-\frac{z}{a})\zeta_n - \zeta_n\cos(1-\frac{z}{a})\zeta_n\}}{\{2+a(\lambda_1+\lambda_2)\}\zeta_n\sin\zeta_n + \{a^2\lambda_1\lambda_2 + a(\lambda_1+\lambda_2) + \zeta_n^2\}\cos\zeta_n}$$

$$\cdot\int_0^t e^{-\frac{k\tau\zeta_n^2}{a^2}}\cdot e^{-\frac{(x-\xi)^2}{4k\tau}}\cdot\left\{e^{-\frac{(y-\eta)^2}{4k\tau}} - \delta\, e^{-\frac{(y+\eta)^2}{4k\tau}}\right\}\frac{d\tau}{\tau}\int_{-\infty}^{\infty}d\xi\int_0^{\infty}u_3(\xi,\eta,0;t-\tau)\,d\eta .$$

where $\delta = 1$ and where the summation extends over the roots of the transcendental equation

$$(a^2 h_1 h_2 + \zeta^2) \tan \zeta - a(h_1 - h_2)\zeta = 0 .$$

In entirely similar manner the solution $u_2(x,y,z;t)$ may be obtained from the solution $u(x,y,z;t)$ in Part III, Case 6 by substitutions identical with those for $u_1(x,y,z;t)$ above. Thus:

$$u_2(x,y,z;t) = \frac{h_2}{2\pi a} \sum_{m=1}^{\infty} \frac{\zeta_m \{\zeta_m \cos \frac{z}{a}\zeta_m + a h_1 \sin \frac{z}{a}\zeta_m\}}{\{2 + a(h_1 - h_2)\}\zeta_m \sin \zeta_m + \{a^2 h_1 h_2 + a(h_2 - h_1) + \zeta_m^2\} \cos \zeta_m}$$

$$\cdot \int_0^t e^{-\frac{k\tau\zeta_m^2}{a^2}} \cdot e^{-\frac{(x-\xi)^2}{4k\tau}} \cdot \left\{ e^{-\frac{(y-\eta)^2}{4k\tau}} - \delta e^{-\frac{(y+\eta)^2}{4k\tau}} \right\} \frac{d\tau}{\tau} \int_{-\infty}^{\infty} d\xi \int_0^{\infty} u_3(\xi,\eta,a;t-\tau)\, d\eta$$

where $\delta = 1$ and the summation extends over the roots of the above transcendental equation.

If in the above expressions of u_1 and u_2 and in the transcendental equation we put $h_2 = 0$ or $h_2 = \infty$ we obtain the solution appropriate to the cases where the boundary $z=a$ is impervious to heat, or is kept at 0°C. Similarly if we put $h_1 = 0$ or $h_1 = \infty$ we obtain the solutions appropriate to the cases where the boundary $z=0$ is impervious to heat, or is kept at 0°C.

For the derivation of $v(x,y,z;t)$ we put $v = v_1 + v_2$ where

$$\lim_{t \to 0} v_1(x,y,z;t) = f(x,y,z)$$

$$\frac{\partial v_1}{\partial z} = 0 \text{ for } z = 0 \text{ and } z = a$$

$$v_1(x,0,z;t) = 0$$

$$\lim_{t \to 0} v_2(x,y,z;t) = 0$$

$$\frac{\partial v_2}{\partial z} - h_1 v_2 = h v_1(x,y,0;t) \quad \text{for } z = 0$$

$$\frac{\partial v_2}{\partial z} - h_2 v_2 = h v_1(x,y,a;t) \quad \text{for } z = a .$$

The expression for $v_1(x,y,z;t)$ may be derived from that of $v(x,y,z;t)$ in Case 1 above by replacing $\sin \frac{m\pi z}{a}$ and $\sin \frac{m\pi\zeta}{a}$ by $\cos \frac{m\pi z}{a}$ and $\cos \frac{m\pi r}{a}$ respectively. Thus:

$$v_1(x,y,z;t) = \frac{1}{2\pi a k t} \sum_{m=1}^{\infty} \cos \frac{m\pi z}{a} \cdot e^{-\frac{k m^2 \pi^2 t}{a^2}}$$

$$\cdot \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} d\xi \int_{c}^{\infty} \left\{ e^{-\frac{(y-\eta)^2}{4kt}} - \delta e^{-\frac{(y+\eta)^2}{4kt}} \right\} d\eta \int_{c}^{a} f(\xi,\eta,\zeta) \sin \frac{m\pi\zeta}{a} d\zeta$$

where $\delta = 1$.

Comparison between the boundary conditions satisfied by $V_2(x,y,z;t)$ and those satisfied by u_1 and u_2 above, leads to the conclusion that

$$v_2(x,y,z;t) = \bar{u}_1(x,y,z;t) + \bar{u}_2(x,y,z;t)$$

where $\bar{u}_1$ is obtained from u_1 by replacing $u_3(x,y,0,t)$ by $v_1(x,y,0,t)$ and where $\bar{u}_2$ is obtained from u_2 by replacing $u_3(x,y,a;t)$ by $v_1(x,y,a;t)$ where $v_1(x,y,z;t)$ is given above.

Case 4.

Temperature gradient $\varphi(x,z;t)$ on $y=0$; boundaries $z=0$ and $z=a$ kept at $0°C$. Initial temperature $f(x,y,z)$.

By analogy with the developments in Case 1 the Laplace transform of the solution $U(x,y,z;t)$ is

$$u^*(x,y,z;p) = -\frac{2}{\pi a} \sum_{m=1}^{\infty} \sin \frac{m\pi z}{a} \int_0^{\infty} d\alpha \int_0^{a} \sin \frac{m\pi\zeta}{a} d\zeta$$

$$\cdot \int_{-\infty}^{\infty} \frac{e^{-\beta_m y}}{\beta_m} \varphi^*(\xi,\zeta,p) \cos \alpha(x-\xi)\, d\xi$$

where $\beta_m = \sqrt{\frac{p}{k} + \alpha^2 + \gamma_m^2}$ and $\gamma_m = \frac{m\pi}{a}$. From the last equation we get

$$u(x,y,z;t) = -\frac{2}{\pi a} \sum_{m=1}^{\infty} \sin \frac{m\pi z}{a} \int_0^{\infty} d\alpha \int_0^{a} \sin \frac{m\pi\zeta}{a} d\zeta$$

$$\cdot \int_{-\infty}^{\infty} \cos \alpha(x-\xi)\, d\xi \int_0^{t} \varphi(\xi,\zeta;t-\tau)\, \psi(y,\tau,\alpha,\gamma_m)\, d\tau$$

where ψ is obtained from the inversion of

$$\psi^*(y;p,\alpha,\gamma_n) = \frac{e^{-\sqrt{\frac{p}{\lambda}+\alpha^2+\gamma_n^2}}}{\sqrt{\frac{p}{\lambda}+\alpha^2+\gamma_n^2}} .$$

By analogy with the expression for the function ψ of Part II, Case 6 we may write at once

$$\psi(y;t,\alpha,\gamma_n) = \frac{\sqrt{\lambda}}{\sqrt{\pi t}} \cdot e^{-\lambda(\alpha^2+\gamma_n^2)t} \cdot e^{-\frac{y^2}{4\lambda t}}$$

whence ultimately

$$u(x,y,z;t) = -\frac{1}{\pi a}\sum_{n=1}^{\infty} \sin\frac{n\pi z}{a}\int_0^a \sin\frac{n\pi\zeta}{a}\,d\zeta$$
$$\int_{-\infty}^{\infty} d\xi \int_0^t e^{-\frac{(x-\xi)^2+y^2}{4\lambda\tau}} \cdot e^{-\frac{\lambda n^2\pi^2\tau}{a^2}} \varphi(\xi,\zeta;t-\tau)\,\tau^{-1}\,d\tau .$$

The solution $v(x,y,z;t)$ is identical with the solution $v_1(x,y,z;t)$ of Case 2.

If in the above expressions for u and v we replace $\sin\frac{n\pi z}{a}$ and $\sin\frac{n\pi\zeta}{a}$ by $\cos\frac{n\pi z}{a}$ and $\cos\frac{n\pi\zeta}{a}$ respectively we obtain the solution appropriate to the case where the boundaries $z=0$ and $z=a$ are impervious to heat. Similarly if we replace $\sin\frac{n\pi z}{a}$ and $\sin\frac{n\pi\zeta}{a}$ by $\sin\frac{(2n+1)\pi z}{2a}$ and $\sin\frac{(2n+1)\pi\zeta}{2a}$ we obtain the solutions appropriate to the case where the boundary $z=0$ is kept at 0°C while the boundary $z=a$ is impervious to heat.

Case 5.

Temperature gradient $\varphi(x,z;t)$ on $y=0$; boundaries $z=0$ and $z=a$ radiating into a medium at 0°C. Initial temperature $f(x,y,z)$.

The derivation of the "u" and "v" solutions is similar to that of Part II, Case 6. For the derivation of the "u" solution, we put $u = u_1 + u_2 + u_3$ where

$$\frac{\partial u_1}{\partial y} = \varphi(x,z;t) \quad \text{for} \quad y=0$$

$$\frac{\partial u_2}{\partial z} = 0 \quad \text{for} \quad z=0 \text{ and } z=a$$

$$\frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} = 0 \quad \text{for} \quad y=0$$

$$\frac{\partial u_1}{\partial z} - h u_1 = \begin{cases} h u_3 & \text{for } z = 0 \\ 0 & \text{for } z = a \end{cases}$$

$$\frac{\partial u_2}{\partial z} + h u_2 = \begin{cases} 0 & \text{for } z = 0 \\ - h u_3 & \text{for } z = a \end{cases} .$$

The Laplace transform of u_3 may be obtained from that of the previous case by replacing sin $\frac{m\pi z}{a}$ and sin $\frac{m\pi\zeta}{a}$ by cos $\frac{m\pi z}{a}$ and cos $\frac{m\pi\zeta}{a}$ respectively. This leads ultimately to

$$u_3(x,y,z;t) = -\frac{1}{\pi a}\sum_{m=1}^{\infty} \cos\frac{m\pi z}{a}\int_0^a \cos\frac{m\pi\zeta}{a}\,d\zeta$$

$$\int_{-\infty}^{\infty} d\xi \int_0^t e^{-\frac{(x-\xi)^2+y^2}{4\kappa\tau}} \cdot e^{-\frac{\kappa m^2\pi^2\tau}{a^2}} \varphi(\xi,\zeta;t-\tau)\,\tau^{-1}\,d\tau .$$

The solutions u_1 and u_2 are formally identical with the corresponding solutions of Case 3, except that now $\delta = -1$, $h_1 = h$, $h_2 = -h$ and u_3 is given by the above equation.

The solution $v(x,y,z;t)$ is given by $v = v_1 + v_2$ where v_1 is obtained from the corresponding solution of Case 3 by putting $\delta = -1$ and where v_2 is obtained in identical manner as that described in Case 3.

Part V. Heat Conduction in the Domain D_3"'

$$(0 < x < \infty,\ 0 < y < \infty,\ 0 < z < a)$$

If in the solutions of the problems in Part IV we replace $\int_{-\infty}^{\infty} d\xi$ by $\int_0^{\infty} d\xi$ and the factor $e^{-\frac{(x-\xi)^2}{4\kappa t}}$ by $e^{-\frac{(x-\xi)^2}{4\kappa t}} \pm e^{-\frac{(x+\xi)^2}{4\kappa t}}$ we obtain at once the solutions of corresponding problems for the domain under consideration for the cases where the boundary $x=0$ is impervious to heat, or kept at 0°C.

For this reason and because of the geometric "similarity" between the boundaries $x=0$ and $y=0$ it will suffice to confine ourselves to the discussion of problems in which the boundaries $x=0$ and $y=0$ are neither impervious to heat nor kept at 0°C. As heretofore, it will suffice to confine ourselves to the discussion of problems involving a single nonhomogeneous boundary condition.

Case 1.

Boundary $y=0$ kept at temperature $\varphi(x,z;t)$; boundary $x=0$ radiating into a medium at 0°C; boundaries $z=0$ and $z=a$ kept at 0°C. Initial temperature $f(x,y,z)$.

Derivation of the "u" solution.

In order to obtain the solution u, we put $u = u_1 + u_2$ where

$$u_1(x,0,z;t) = \varphi(x,z;t)$$

$$u_1(x,y,0;t) = u_1(x,y,a;t) = 0$$

$$\frac{\partial u_1}{\partial x} = 0 \quad \text{for} \quad x=0$$

$$u_2(x,0,z;t) = 0$$

$$u_2(x,y,0;t) = u_2(x,y,a;t) = 0$$

$$\frac{\partial u_2}{\partial x} = h(u_2+u_1) \qquad \text{for} \quad x=0 \; .$$

The solution u_1 may be obtained from the solution u of Part IV, Case 1 by assuming $\varphi(-x,z) = \varphi(x,z)$. We thus get

$$u_1(x,y,z;t) = \frac{y}{\pi a k} \sum_{n=1}^{\infty} \sin\frac{n\pi z}{a} \int_0^a \sin\frac{n\pi\zeta}{a}\, d\zeta \int_0^{\infty} d\xi$$

$$\cdot \int_0^t e^{-\frac{y^2}{4k\tau}} \cdot e^{-\frac{kn^2\pi^2\tau}{a^2}} \varphi(\xi,\zeta;t-\tau)\,\tau^{-2}\,d\tau \; .$$

The solution u_2 may be obtained from the solution u of Part IV, Case 2 in the following manner:

a. Interchange x and y and replace ξ by η and $\int_{-\infty}^{\infty} d\xi$ by $\int_0^{\infty} d\eta$.

b. In the expression thus obtained replace the factor $e^{-\frac{(y-\eta)^2}{4k\tau}}$ by $e^{-\frac{(y-\eta)^2}{4k\tau}} - e^{-\frac{(y+\eta)^2}{4k\tau}}$ and the function $\varphi(x,z;t)$ by $-u_1(0,y,z,t)$. In this manner we get

$$u_2(x,y,z;t) = -\frac{h}{2\pi a k} \sum_{n=1}^{\infty} \sin\frac{n\pi z}{a} \int_0^a \sin\frac{n\pi\zeta}{a}\, d\zeta \int_0^{\infty} d\eta$$

$$\cdot \int_0^t \left\{ e^{-\frac{(y-\eta)^2}{4k\tau}} + \delta e^{-\frac{(y+\eta)^2}{4k\tau}} \right\} e^{-\frac{kn^2\pi^2\tau}{a^2}} u_1(0,\xi,\zeta;t-\tau)\,\tau^{-2}\,d\tau$$

$$\cdot \int_0^{\infty} e^{-h\rho} (x+\rho)\, e^{-\frac{(x+\rho)^2}{4k\tau}}\, d\rho$$

where $\delta = -1$.

Derivation of solution $v(x,y,z;t)$

In order to obtain v, we put $v = v_1 + v_2$ where

$$\lim_{t \to 0} v_1(x,y,z;t) = f(x,y,z)$$

$$v_1 = 0 \quad \text{for} \quad y=0,\ z=0 \ \text{and} \ z=a$$

$$\frac{\partial v_1}{\partial x} = 0 \qquad \text{for} \quad x=0$$

$$\lim_{t \to 0} v_2(x,y,z;t) = 0$$

$$v_2 = 0 \quad \text{for} \quad y=0,\ z=0 \ \text{and} \ z=a$$

$$\frac{\partial v_2}{\partial x} = h(v_2 + v_1) \qquad \text{for} \quad x=0 \ .$$

The solution v_1 may be obtained from the corresponding solution of Part IV, Case 2 by interchanging x and y and by replacing in the expression thus obtained ξ by η, $\int_{-\infty}^{\infty} d\xi$ by $\int_0^{\infty} d\eta$ and the factor $e^{-\frac{(y-\eta)^2}{4\kappa t}}$ by $e^{-\frac{(y-\eta)^2}{4\kappa t}} - e^{-\frac{(y+\eta)^2}{4\kappa t}}$.

In this manner we get

$$v_1(x,y,z;t) = \frac{1}{2\pi a \kappa t} \sum_{n=1}^{\infty} \sin\frac{n\pi z}{a} \cdot e^{-\frac{\kappa n^2 \pi^2 t}{a^2}} \ .$$

$$\cdot \int_0^{\infty} \left\{ e^{-\frac{(y-\eta)^2}{4\kappa t}} + \delta e^{-\frac{(y+\eta)^2}{4\kappa t}} \right\} d\eta \int_0^{\infty} \left\{ e^{-\frac{(x-\xi)^2}{4\kappa t}} + e^{-\frac{(x+\xi)^2}{4\kappa t}} \cdot \right\} d\xi$$

$$\cdot \int_0^{a} f(\xi,\eta,\zeta) \sin\frac{n\pi\zeta}{a} \, d\zeta$$

with $\delta = -1$.

The function v_2 may be obtained from the expression for u_1 by replacing $u_1(0,\xi,\zeta;t-\tau)$ by $v_1(0,\xi,\zeta;t-\tau)$ where v_1 is given above.

If in the above derivations we interchange x and y we obtain the solutions appropriate to the case where the boundary x=0 is kept at temperature $\varphi(y,z;t)$. The boundary y=0 radiates into a medium at 0°C and the boundaries z=0 and z=a are kept at 0°C.

Case 2.

Boundary $x=0$ radiates into a medium at $0°C$; temperature gradient $\varphi(x,z;t)$ on $y=0$; boundaries $z=0$ and $z=a$ kept at $0°C$. Initial temperature $f(x,y,z)$.

In order to derive the "u" solution, we put $u = u_1 + u_2$ where

$$\frac{\partial u_1}{\partial x} = 0 \qquad \text{for } x = 0$$

$$\frac{\partial u_1}{\partial y} = \varphi(x,z;t) \qquad \text{for } y = 0$$

$$u_1(x,y,0;t) = u_1(x,y,a;t) = 0$$

$$\frac{\partial u_2}{\partial x} = h(u_1 + u_2) \qquad \text{for } x = 0$$

$$\frac{\partial u_2}{\partial y} = 0 \qquad \text{for } y = 0$$

$$u_2(x,y,0;t) = u_2(x,y,a;t) = 0 .$$

The solution u_1 may be obtained from the solution u of Part IV, Case 1 by replacing $\int_{-\infty}^{\infty} d\xi$ by $\int_0^{\infty} d\xi$ and the factor $e^{-\frac{(x-\xi)^2}{4kt}}$ by $e^{-\frac{(x-\xi)^2}{4kt}} + e^{-\frac{(x+\xi)^2}{4kt}}$.

Thus

$$u_1(x,y,z;t) = -\frac{1}{\pi a} \sum_{n=1}^{\infty} \sin\frac{n\pi z}{a} \int_0^a \sin\frac{n\pi\zeta}{a}\, d\zeta \int_0^{\infty} d\xi$$

$$\int_0^t e^{-\frac{y^2}{4k\tau}} \left\{ e^{-\frac{(x-\xi)^2}{4k\tau}} + e^{-\frac{(x+\xi)^2}{4k\tau}} \right\} e^{-\frac{kn^2\pi^2\tau}{a^2}}\, \varphi(\xi,\zeta,t-\tau)\, \tau^{-1}\, d\tau .$$

The solution u_2 is formally identical with the solution u_2 of the previous case, except that now $\delta = 1$.

In order to obtain the "v" solution we put $v = v_1 + v_2$ where

$$\lim_{t \to 0} v_1(x,y,z;t) = f(x,y,z)$$

$$\frac{\partial v_1}{\partial x} = 0 \qquad \text{for } x = 0$$

$$\frac{\partial v_1}{\partial y} = 0 \qquad \text{for } y = 0$$

$$v_1 = 0 \quad \text{for } z = 0 \text{ and } z = a$$

$$\lim_{t \to 0} v_2(x,y,z;t) = 0$$

Case 2.

Boundary $x=0$ radiates into a medium at 0°C; temperature gradient $\varphi(x,z;t)$ on $y=0$; boundaries $z=0$ and $z=a$ kept at 0°C. Initial temperature $f(x,y,z)$.

In order to derive the "u" solution, we put $u = u_1 + u_2$ where

$$\frac{\partial u_1}{\partial x} = 0 \qquad \text{for } x = 0$$

$$\frac{\partial u_1}{\partial y} = \varphi(x,y;t) \qquad \text{for } y = 0$$

$$u_1(x,y,0;t) = u_1(x,y,a;t) = 0$$

$$\frac{\partial u_2}{\partial x} = h(u_2 + u_1) \qquad \text{for } x = 0$$

$$\frac{\partial u_2}{\partial y} = 0 \qquad \text{for } y = 0$$

$$u_2(x,y,0;t) = u_2(x,y,a;t) = 0 \; .$$

The solution u_1 may be obtained from the solution u of Part IV, Case 4 by replacing $\int_{-\infty}^{\infty} d\xi$ by $\int_0^{\infty} d\xi$ and the factor $e^{-\frac{(x-\xi)^2}{4\kappa t}}$ by $e^{-\frac{(x-\xi)^2}{4\kappa t}} + e^{-\frac{(x+\xi)^2}{4\kappa t}}$.

Thus

$$u_1(x,y,z;t) = -\frac{1}{\pi a}\sum_{n=1}^{\infty} \sin\frac{n\pi z}{a}\int_0^a \sin\frac{n\pi\zeta}{a}\,d\zeta\int_0^{\infty} d\xi$$

$$\int_0^t e^{-\frac{y^2}{4\kappa\tau}}\left\{e^{-\frac{(x-\xi)^2}{4\kappa\tau}} + e^{-\frac{(x+\xi)^2}{4\kappa\tau}}\right\}\cdot e^{-\frac{\kappa n^2\pi^2\tau}{a^2}}\,\varphi(\xi,\zeta;t-\tau)\,\tau^{-1}\,d\tau \; .$$

The solution u_2 is formally identical with the solution u_2 of the previous case, except that now $\delta = 1$.

In order to obtain the "v" solution we put $v = v_1 + v_2$ where

$$\lim_{t\to 0} v_1(x,y,z;t) = f(x,y,z)$$

$$\frac{\partial v_1}{\partial x} = 0 \qquad \text{for } x = 0$$

$$\frac{\partial v_1}{\partial y} = 0 \qquad \text{for } y = 0$$

$$v_1 = 0 \quad \text{for } z = 0 \text{ and } z = a$$

$$\lim_{t\to 0} v_2(x,y,z;t) = 0$$

$$\frac{\partial v_1}{\partial y} = 0 \qquad \text{for} \quad y = 0$$

$$v_1 = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad z = a$$

$$\frac{\partial v_1}{\partial x} = \lambda(v_1 + c_1) \qquad \text{for} \quad x = 0 \quad .$$

The expression for v_1 is identical with that of the previous case, except that now $\delta = 1$. The solution v_2 may be obtained from the solution u_2 by replacing $u_1(0,\xi,\zeta;t-\tau)$ by $v_1(0,\xi,\zeta;t-\tau)$.

If in the derivations under Case 1 and Case 2 we replace $\sin\frac{n\pi z}{a}$ and $\sin\frac{n\pi\zeta}{a}$ by $\cos\frac{n\pi z}{a}$ and $\cos\frac{n\pi\zeta}{a}$ we obtain the solutions appropriate to the cases where the boundaries z=0 and z=a are impervious to heat, the other boundary conditions being the same as before. Similarly if $\sin\frac{n\pi z}{a}$ and $\sin\frac{n\pi\zeta}{a}$ are replaced by $\sin\frac{(2n+1)\pi z}{2a}$ and $\sin\frac{(2n+1)\pi\zeta}{2a}$ we obtain the solutions appropriate to the case where the boundary z=0 is kept at 0°C while the boundary z=a is impervious to heat.

Part VI <u>Heat Condition in the Domain $D_3^{(iv)}$</u> $(0 \le x \le a,\ 0 \le y \le h,\ -\infty < z < \infty)$

Case 1.

Boundary y=0 kept at temperature $\varphi(x,z;t)$; other boundaries kept at 0°C.
Initial temperature $f(x,y,z)$.

<u>Derivation of solution $u(x,y,z;t)$</u>

By analogy with the developments in Part IV, Case 1, the Laplace transform of u is

$$u^*(x,y,z;p) = \frac{2}{\pi a}\sum_{m=1}^{\infty}\sin\frac{m\pi x}{a}\int_0^{\infty}d\gamma\int_0^{a}\sin\frac{m\pi\xi}{a}\,d\xi$$

$$\int_{-\infty}^{\infty}\frac{\sinh\beta_m(h-y)}{\sinh\beta_m h}\,\varphi^*(\xi,\zeta;p)\cos\gamma(z-\zeta)\,d\zeta$$

where $\beta_m = \sqrt{\frac{p}{k} + \alpha_m^2 + \gamma^2}$ and $\alpha_m = \frac{m\pi}{a}$.

The last equation yields in the usual manner

$$u(x,y,z;t) = \frac{2}{\pi a}\sum_{m=1}^{\infty}\sin\frac{m\pi x}{a}\int_0^{\infty}d\gamma\int_0^a \sin\frac{m\pi\xi}{a}\,d\xi \tag{A}$$

$$\int_{-\infty}^{\infty}\cos\gamma(z-\zeta)\,d\zeta\int_0^t \varphi(\xi,\zeta;t-\tau)\;\psi(y;\tau,\alpha_m,\gamma)\,d\tau$$

where the function ψ is obtained from the inversion of

$$\int_0^{\infty} e^{-pt}\,\psi(y;t,\alpha_m,\gamma)\,dt = \frac{\sinh\beta_m(\ell-y)}{\sinh\beta_m\ell}.$$

In view of the developments in Part I, Case 1 (equation (16), (19) and (24)) we can write at once

$$\psi(y;t,\alpha_m,\gamma) = e^{-k(\alpha_m^2+\gamma^2)t}\cdot\Phi(y;t), \qquad \text{where}$$

$$\Phi(y;t) = \frac{2\pi k}{\ell^2}\sum_{m=1}^{\infty} m\sin\frac{m\pi y}{\ell}\cdot e^{-\frac{km^2\pi^2 t}{\ell^2}}.$$

With the aid of the last two equations and of the identity (20) we ultimately get

$$u(x,y,z;t) = \frac{2\sqrt{\pi k}}{a\ell^2}\sum_{m=1}^{\infty}\sum_{m=1}^{\infty} m\sin\frac{m\pi x}{a}\sin\frac{m\pi y}{\ell}$$

$$\int_0^a \sin\frac{m\pi\xi}{a}\,d\xi\int_{-\infty}^{\infty}d\zeta\int_0^t \varphi(\xi,\zeta;t-\tau)\cdot e^{-\frac{(z-\zeta)^2}{4k\tau}}$$

$$\cdot\; e^{-\left(\frac{km^2\pi^2\tau}{a^2}+\frac{km^2\pi^2\tau}{\ell^2}\right)}\cdot\frac{d\tau}{\sqrt{\tau}}.$$

Derivation of solution $v(x,y,z;t)$

As in Part III, Case 1 we have

$$v(x,y,z;t) = \frac{1}{\pi^3}\iiint_{-\infty}^{\infty}\Phi(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta$$

$$\cdot\iiint_0^{\infty} e^{-k(\alpha^2+\beta^2+\gamma^2)t}\cos\alpha(x-\xi)\cos\beta(y-\eta)\cos\gamma(z-\zeta)\,d\alpha\,d\beta\,d\gamma$$

where by analogy with the developments there the function ϕ must satisfy the conditions

$$\left.\begin{aligned} \phi(x+2na, y, z) &= f(x,y,z) \\ \phi(-x+2na, y, z) &= -f(x,y,z) \end{aligned}\right\} \qquad \begin{aligned} &0 \le x \le a \\ &n = \pm 1, \pm 2, \pm 3, \ldots \pm\infty \end{aligned}$$

$$\left.\begin{aligned} \phi(x, y+2mb, z) &= f(x,y,z) \\ \phi(x, -y+2mb, z) &= -f(x,y,z) \end{aligned}\right\} \qquad \begin{aligned} &0 \le y \le b \\ &m = \pm 1, \pm 2, \pm 3, \ldots \pm\infty \, . \end{aligned}$$

With the aid of identity (20) and of the last four equations, the solution v(x,y,z;t) becomes

$$v(x,y,z;t) = \frac{1}{8(\pi k t)^{3/2}} \int_{-\infty}^{\infty} e^{-\frac{(z-\zeta)^2}{4kt}} \, d\zeta$$

$$\cdot \int_0^a \left\{ \sum_{n=-\infty}^{\infty} \left[e^{-\frac{(x-\xi-2na)^2}{4kt}} - e^{-\frac{(x+\xi-2na)^2}{4kt}} \right] \right\} d\xi$$

$$\cdot \int_0^b F(\xi,\eta,\zeta) \left\{ \sum_{m=-\infty}^{\infty} \left[e^{-\frac{(y-\eta-2mb)^2}{4kt}} - e^{-\frac{(y+\eta-2mb)^2}{4kt}} \right] \right\} d\eta$$

Making use of the identities (36) and (36') the last equation ultimately becomes

$$v(x,y,z;t) = \frac{2}{ab\sqrt{\pi k t}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}$$

$$\cdot \int_0^a \sin\frac{n\pi\xi}{a} \, d\xi \int_0^b \sin\frac{m\pi\eta}{b} \, d\eta \int_{-\infty}^{\infty} f(\xi,\eta,\zeta) \, e^{-\frac{(z-\zeta)^2}{4kt}} \, d\zeta$$

Case 2.

Temperature gradient φ (x,z;t) on boundary y=b; other boundaries kept 0°C. Initial temperature f(x,y,z).

<u>Derivation of solution u(x,y,z;t)</u>

The counterpart of the expression for u* in Case 1 is

$$u^*(x,y,z;p) = \frac{2}{\pi a} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{a} \int_0^{\infty} d\gamma \int_0^a \sin\frac{n\pi\xi}{a} \, d\xi \, \cdot$$

$$\cdot \int_{-\infty}^{\infty} \frac{\sinh \beta_n y}{\beta_n \cosh \beta_n b} \cdot \varphi^*(\xi,\zeta;p) \cos\gamma(z-\zeta) \, d\zeta$$

where $\quad \beta_m = \sqrt{\frac{p}{k} + \alpha_m^2 + \gamma^2}$.

The solution u(x,y,z;t) is formally given by equation (A) of Case 1 where now ψ is obtained from the inversion of

$$\int_0^\infty e^{-p\tau}\, \psi(y;\tau,\alpha_m,\gamma)\, d\tau = \frac{\sinh \beta_m y}{\beta_m \cosh \beta_m b} .$$

As in Part III Case 2 we have

$$\psi(y;\tau,\alpha_m,\gamma) = \frac{2k}{b}\, e^{-k(\alpha_m^2+\gamma^2)\tau} \cdot \sum_{m=0}^{\infty} (-1)^m\, e^{-\frac{k(2m+1)^2\pi^2\tau}{4b^2}} \sin\frac{(2m+1)\pi y}{2b} .$$

Making use of the above expression for ψ and of identity (20), formula (A) of Case 1 ultimately yields

$$u(x,y,z;t) = \frac{2\sqrt{k}}{ab\sqrt{\pi}} \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} (-1)^m \sin\frac{n\pi x}{a} \sin\frac{(2m+1)\pi y}{2b}$$

$$\cdot \int_0^a \sin\frac{n\pi\xi}{a}\, d\xi \int_{-\infty}^{\infty} d\zeta \int_0^t \varphi(\xi,\zeta;t-\tau)\cdot e^{-\frac{(z-\zeta)^2}{4k\tau}} \cdot e^{-\frac{kn^2\pi^2\tau}{a^2}} \cdot$$

$$\cdot\, e^{-\frac{k(2m+1)^2\pi^2\tau}{4b^2}}\, \tau^{-\frac{1}{2}}\, d\tau .$$

Derivation of solution v(x,y,z;t)

As in the problem in Part III Case 2, the desired solution is identical with that appropriate to a domain identical with that of the previous case, except that now y ranges from 0 to 2b, the two bounding planes y=0 and y=2b as well as the boundaries x=0 and x=a being kept at 0°C, the initial temperature Φ (x,y,z) being defined as follows:

$$\Phi(x,y,z) = f(x,y,z) \qquad \text{for} \qquad 0 < y < b$$

$$\Phi(x,2b-y,z) = f(x,y,z) \qquad \text{for} \qquad b < y < 2b .$$

Starting with the expression v(x,y,z;t) of the previous case, we ultimately get:

$$u(x,y,z;t) = \frac{2}{ab\sqrt{\pi k t}} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sin \frac{n\pi x}{a} \sin \frac{(2m+1)\pi y}{2b}$$

$$\int_0^a \sin \frac{n\pi\xi}{a}\, d\xi \int_0^b \sin \frac{(2m+1)\pi\eta}{2b}\, d\eta \int_{-\infty}^{\infty} f(\xi,\eta,\zeta)\, e^{-\frac{(z-\zeta)^2}{4kt}}\, d\zeta \ .$$

Case 3.

Boundary y=0 radiates into a medium at temperature φ (x,z;t); other boundaries kept at 0°C. Initial temperature f(x,y,z).

<u>Derivation of solution u(x,y,z;t)</u>

By analogy with the developments in the previous cases and in Part III Case 3, the desired solution has for its Laplace transform the expression

$$u^*(x,y,z;p) = \frac{2h}{\pi a} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \int_0^{\infty} d\gamma \int_0^a \sin \frac{n\pi\xi}{a}\, d\xi$$

$$\cdot \int_{-\infty}^{\infty} \frac{\sinh \beta_n (z-b)}{\beta_n \cosh \beta_n b + h \sinh \beta_n b}\ \varphi^*(\xi,\eta;p) \cos \gamma (z-\zeta)\, d\zeta$$

where $\beta_n = \sqrt{\frac{p}{k} + \alpha_n^2 + \gamma^2}$ and $\alpha_n = \frac{n\pi}{a}$.

The inversion of the last equation yields formula (A) of Case 1 where now ψ is obtained from the inversion of

$$\int_0^{\infty} e^{-pt}\, \psi(y;t,\alpha_n,\gamma)\, dt = \frac{h \sinh \beta_n (z-b)}{\beta_n \cosh \beta_n b + h \sinh \beta_n b} \ .$$

The counterpart of the formula for ψ in Part III Case 3 is:

$$\psi(y;t,\alpha_n,\gamma) = 2hk\, e^{-k(\alpha_n^2+\gamma^2)t}$$

$$\cdot \sum_{m=1}^{\infty} \frac{\zeta_m \sin\left(1-\frac{y}{b}\right)\zeta_m\, e^{-\frac{kt\zeta_m^2}{b^2}}}{\{(1+bh)bh + \zeta_m^2\} \cos \zeta_m} \ .$$

Substituting the above expression of ψ in formula (A) and making use of identity (20) the desired solution becomes

$$u(x,y,z;t) = \frac{2h\sqrt{k}}{a\sqrt{\pi}} \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \sin\frac{m\pi x}{a} \frac{\zeta_m \sin\left(1-\frac{y}{b}\right)\zeta_m}{\{(1+bh)bh+\zeta_m^2\}\cos\zeta_m}$$

$$\cdot \int_0^a \sin\frac{m\pi\xi}{a}\, d\xi \int_{-\infty}^{\infty} d\zeta \int_0^t \varphi(\xi,\zeta;t-\tau)\, e^{-\frac{k\tau\zeta_m^2}{b^2}} \cdot e^{-\frac{km^2\pi^2\tau}{a^2}}\, e^{-\frac{(z-\zeta)^2}{4k\tau}} \cdot \frac{d\tau}{\sqrt{\tau}}$$

where the ζ_m's are the roots of the transcendental equation

$$\zeta + bh \tan\zeta = 0 .$$

Derivation of solution v(x,y,z;t)

We put $v = v_1 + v_2$ where

$$\lim_{t\to 0} v_1(x,y,z;t) = f(x,y,z)$$

$$v_1(x,b,z;t) = 0$$

$$\frac{\partial v_1}{\partial y} = 0 \quad \text{for } y = 0$$

$$\lim_{t\to 0} v_2(x,y,z;t) = 0$$

$$v_2(x,b,z;t) = 0$$

$$\frac{\partial v_2}{\partial y} = h(v_2+v_1) \quad \text{for } y = 0 .$$

The solution $v_1(x,y,z;t)$ is identical with the "v" solution of Case 1 provided that y now ranges from -b to b and the initial temperature Φ (x,y,z) satisfies the conditions

$$\left.\begin{aligned} \Phi(x+2ma,y,z) &= f(x,y,z) \\ \Phi(-x+2ma,y,z) &= -f(x,y,z) \end{aligned}\right\} \qquad 0 \le x \le a, \quad m = \pm1, \pm2, \pm3, \ldots \pm\infty$$

$$\left.\begin{aligned} \Phi(x,\pm y+4mb,z) &= f(x,y,z) \\ \Phi(x,\pm y+4mb+b,z) &= -f(x,y,z) \end{aligned}\right\} \qquad 0 \le y \le b, \quad m = \pm1, \pm2, \pm3, \ldots \pm\infty .$$

(See also Part III, Case 3)

Proceeding as in Case 1, we get

$$v_1(x,y,z;t) = \frac{1}{8(\pi k t)^{3/2}} \int_{-\infty}^{\infty} e^{-\frac{(z-\zeta)^2}{4\kappa t}} d\zeta$$

$$\cdot \int_0^a \left\{ \sum_{m=-\infty}^{\infty} \left[e^{-\frac{(x-\xi-2ma)^2}{4\kappa t}} - e^{-\frac{(x+\xi-2ma)^2}{4\kappa t}} \right] \right\} d\xi$$

$$\cdot \int_0^b f(\xi,\eta,\zeta) \left\{ \sum_{m=-\infty}^{\infty} \left[e^{-\frac{(y+\eta-4mb)^2}{4\kappa t}} + e^{-\frac{(y-\eta-4mb)^2}{4\kappa t}} \right.\right.$$

$$\left.\left. - e^{-\frac{(y+\eta-4mb-2a)^2}{4\kappa t}} - e^{-\frac{(y-\eta-4mb-2a)^2}{4\kappa t}} \right] \right\} d\eta .$$

Making use of (36) and (36') the last equation becomes

$$v_1(x,y,z;t) = \frac{2}{ab\sqrt{\pi k t}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin\frac{n\pi x}{a} \cos\frac{(2m+1)\pi y}{2b}$$

$$\cdot \int_0^a \sin\frac{n\pi\xi}{a} d\xi \int_0^b \cos\frac{(2m+1)\pi\eta}{2b} d\eta \int_{-\infty}^{\infty} f(\xi,\eta,\zeta)\, e^{-\frac{(z-\zeta)^2}{4\kappa t}} d\zeta .$$

The function $v_2(x,y,z;t)$ may be obtained from $u(x,y,z;t)$ by replacing $\varphi(x,z;t)$ by $-v_1(x,0,z;t)$.

Case 4.

Radiation at the boundary y=0 into a medium at temperature $\varphi(x,z;t)$; boundary y=b impervious to heat; boundaries x=0 and x=a kept at 0°C. Initial temperature f(x,y,z).

Derivation of solution u(x,y,z;t)

The solution u(x,y,z;t) is once more given by formula (A) of Case 1 where the function ψ is obtained from the inversion of

$$\int_0^{\infty} e^{-pt} \psi(y;t,\alpha_n,\gamma)\, dt = \frac{h \cosh \beta_n (y-b)}{\beta_n \cosh \beta_n b + h \sinh \beta_n b}$$

where $\beta_n = \sqrt{\frac{p}{k} + \alpha_n^2 + \gamma^2}$ and $\alpha_n = \frac{n\pi}{a}$.

By analogy with the expression for ψ in Part III, Case 4, we have

$$\psi(y; t, \alpha_m, \gamma) = 2h^2 k\, e^{-k(\alpha_n + \gamma^2)t} \cdot \sum_{n=1}^{\infty} \frac{\zeta_n \cos\left(1 - \frac{y}{b}\right)\zeta_n \, e^{-\frac{kt\zeta_n^2}{b^2}}}{\{(1 + bh)\,bh + \zeta_n^2\} \sin \zeta_n}$$

where the ζ_m's are the roots of

$$\zeta \tan \zeta = bh.$$

Substituting ψ in formula (A) and using (20) the desired solution becomes

$$u(x, y, z; t) = \frac{2h\sqrt{k}}{a\sqrt{\pi}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin \frac{n\pi x}{a} \, \frac{\zeta_m \cos\left(1 - \frac{y}{b}\right)\zeta_m}{\{(1 + bh)\,bh + \zeta_m^2\} \sin \zeta_m}$$

$$\cdot \int_0^a \sin \frac{n\pi\xi}{a}\, d\xi \int_{-\infty}^{\infty} d\zeta \int_0^t \varphi(\xi, \zeta; t - \tau)\, e^{-\frac{k\tau\zeta_m^2}{b^2}} \cdot e^{-\frac{kn^2\pi^2\tau}{a^2}} \cdot e^{-\frac{(z-\zeta)^2}{4k\tau}} \cdot \frac{d\tau}{\sqrt{\tau}} .$$

__Derivation of solution v(x,y,z;t)__

We put $v = v_1 + v_2$ where

$$\lim_{t \to 0} v_1(x, y, z; t) = f(x, y, z)$$

$$\frac{\partial v_1}{\partial y} = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad y = b$$

$$v_1 = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = a$$

$$\lim_{t \to 0} v_2(x, y, z; t) = 0$$

$$\frac{\partial v_2}{\partial y} = 0 \quad \text{for} \quad y = b$$

$$\frac{\partial v_2}{\partial y} = h(v_2 + v_1) \quad \text{for} \quad y = 0$$

$$v_2 = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = a .$$

The procedure for deriving v_1 is similar to that for the derivation of v in Case 1, except that the function ϕ must now satisfy the conditions

$$\left.\begin{aligned} \phi(x + 2na, y, z) &= f(x, y, z) \\ \phi(-x + 2na, y, z) &= -f(x, y, z) \end{aligned}\right\} \qquad \begin{aligned} &0 \le x \le a \\ &n = \pm 1, \pm 2, \pm 3, \ldots \pm \infty \end{aligned}$$

$$\left.\begin{aligned} \phi(x, y + 2mb, z) &= f(x, y, z) \\ \phi(x, y + 2mb + b, z) &= f(x, y, z) \end{aligned}\right\} \qquad \begin{aligned} &0 \le y \le b \\ &m = 0, \pm 1, \pm 2, \ldots \pm \infty . \end{aligned}$$

Proceeding as in Case 1, we get

$$v_1(x,y,z;t) = \frac{1}{8(\pi k t)^{3/2}} \int_{-\infty}^{\infty} e^{-\frac{(z-\zeta)^2}{4kt}} d\zeta \int_0^a \left\{ \sum_{m=-\infty}^{\infty} \left[e^{-\frac{(x-\xi-2ma)^2}{4kt}} - e^{-\frac{(x+\xi-2ma)^2}{4kt}} \right] \right\} d\xi$$

$$\cdot \int_0^b f(\xi,\eta,\zeta) \left\{ \sum_{m=-\infty}^{\infty} \left[e^{-\frac{(y-\eta-2mb)^2}{4kt}} + e^{-\frac{(y+\eta-2mb)^2}{4kt}} \right] \right\} d\eta .$$

Making use of the identities (36) and (36') the last equation ultimately becomes

$$v_1(x,y,z;t) = \frac{1}{ab\sqrt{\pi k t}} \sum_{m=1}^{\infty} \sin\frac{m\pi x}{a} \int_0^a \sin\frac{m\pi\xi}{a}\, d\xi .$$

$$\cdot \int_{-\infty}^{\infty} f(\xi,\eta,\zeta)\, e^{-\frac{(z-\zeta)^2}{4kt}} d\zeta + \frac{2}{ab\sqrt{\pi k t}} \sum_{m=1}^{\infty} \cos\frac{m\pi y}{b}$$

$$\cdot \int_0^b \cos\frac{m\pi\eta}{b}\, d\eta \int_{-\infty}^{\infty} f(\xi,\eta,\zeta)\, e^{-\frac{(z-\zeta)^2}{4kt}} d\zeta .$$

The expression for $v_2(x,y,z;t)$ may be obtained from the expression for $u(x,y,z;t)$ by replacing $\varphi(x,z;t)$ by $-v_1(x,0,z;t)$.

Case 5.

Radiation at the boundary $y=0$ into a medium at temperature $\varphi(x,z;t)$; boundary $y=b$ radiating into a medium 0°C; boundaries $x=0$ and $x=a$ kept at 0°C. Initial temperature 0°C.

The solution $u(x,y,z;t)$ is given by formula (A) of Case 1 where ψ is obtained from the inversion of

$$\int_0^{\infty} e^{-pt}\, \psi(y;i,\alpha_m,\gamma)\, dt = \frac{h\{\beta_m \cosh \beta_m(y-b) - h \sinh \beta_m(y-b)\}}{(\beta_m^2 + h^2)\sinh \beta_m b + 2h\beta_m \cosh \beta_m b}$$

where as before $\beta_m = \sqrt{\frac{p}{k} + \alpha_m^2 + \gamma^2}$ and $\alpha_m = \frac{m\pi}{a}$.

(The second member of the last equation is the counterpart of the corresponding expression in Part III Case 5 with $h_1 = h$ and $h_2 = -h$)

By analogy with the expression for ψ in Part III Case 5 we have

$$\psi\,(y;\,t,\,\alpha_m,\,\gamma) = \frac{2hk}{b}\sum_{m=1}^{\infty} e^{-\frac{kt\zeta_m^2}{b^2}}\cdot e^{-k(\alpha_m^2+\gamma^2)t}$$

$$\cdot\frac{\zeta_m\left\{2h\sin\left(\frac{y}{b}-1\right)\zeta_m - \zeta_m\cos\left(\frac{y}{b}-1\right)\zeta_m\right\}}{2(1+2h)\zeta_m\sin\zeta_m + (\zeta_m^2 - b^2h^2 - 2bh)\cos\zeta_m}$$

where the summation is extended over the roots of the transcendental equation

$$(\zeta^2 - b^2h^2)\tan\zeta - 2bh\zeta = 0 \,.$$

Substituting the above expression of ψ in formula (A) of Case 1 and using (20) the desired solution becomes

$$u\,(x,y,z;t) = \frac{2h\sqrt{k}}{ab\sqrt{\pi}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}$$

$$\cdot\frac{\zeta_m\left\{bh\sin\left(\frac{y}{b}-1\right)\zeta_m - \zeta_m\cos\left(\frac{y}{b}-1\right)\zeta_m\right\}}{2(1+bh)\zeta_m\sin\zeta_m + (\zeta_m^2 - b^2h^2 - 2bh)\cos\zeta_m}\int_0^a \sin\frac{n\pi\xi}{a}\,d\xi$$

$$\cdot\int_{-\infty}^{\infty} d\zeta\int_0^t \varphi\,(\xi,\zeta;t-\tau)\cdot e^{-\frac{k\tau\zeta_m^2}{b^2}}\, e^{-\frac{kn^2\pi^2\tau}{a^2}}\, e^{-\frac{(z-\zeta)^2}{4k\tau}}\,\frac{d\tau}{\sqrt{\tau}}$$

where the ζ_m's are the roots of the above transcendental equation.

Case 6.

Radiation at the boundary y=b into a medium at temperature φ (x,z;t); boundary y=0 radiating into a medium at 0°C; boundaries x=0 and x=a kept at 0°C. Initial temperature 0°C.

The solution is given by formula (A) of Case 1 where ψ is obtained from the inversion of

$$\int_0^{\infty} e^{-pt}\,\psi\,(y;\,t,\,\alpha_m,\,\gamma)\,dt = \frac{-h(\beta_m\cosh\beta_m b + h\sinh\beta_m b)}{(\beta_m^2+h^2)\sinh\beta_m b + 2h\beta_m\cosh\beta_m b}\,.$$

(The second member of the last equation is the analogy of the corresponding expression of Part III, Case 6 with h_1 = h and h_2 = -h)

By analogy with the expression for ψ in Part III, Case 6 we have

$$\psi(y;t,\alpha_n,\gamma) = -\frac{2hk}{b}\sum_{m=1}^{\infty} e^{-\frac{kt\zeta_m^2}{b^2}} \cdot e^{-k(\alpha_n^2+\gamma^2)t}$$

$$\cdot \frac{\zeta_m\left\{\zeta_m \cos\frac{y}{b}\zeta_m + bh\sin\frac{y}{b}\zeta_m\right\}}{2(1+bh)\zeta_m \sin\zeta_m + (\zeta_m^2 - b^2h^2 - 2bh)\cos\zeta_m}$$

where the ζ_m's are the roots of

$$(\zeta^2 - b^2h^2)\tan\zeta - 2bh\zeta = 0 \quad .$$

Substituting the above expression of ψ in formula (A) and using (20) the desired solution becomes

$$u(x,y,z;t) = \frac{-2h\sqrt{k}}{ab\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{a}\,\frac{\zeta_m\left\{\zeta_m\cos\frac{y}{b}\zeta_m + bh\sin\frac{y}{b}\zeta_m\right\}}{2(1+bh)\zeta_m\sin\zeta_m + (\zeta_m^2 - b^2h^2 - 2bh)\cos\zeta_m}$$

$$\cdot\int_0^a \sin\frac{n\pi\xi}{a}\,d\xi\int_{-\infty}^{\infty}d\zeta\int_0^t \varphi(\xi,\zeta;t-\tau)\cdot e^{-\frac{k\tau\zeta_m^2}{b^2}}\cdot e^{-\frac{kn^2\pi^2\tau}{a^2}}\cdot e^{-\frac{(z-\zeta)^2}{4k\tau}}\cdot\frac{d\tau}{\sqrt{\tau}}$$

where the ζ_m's are the roots of the above transcendental equation.

Case 7.

Initial temperature $f(x,y,z)$; boundaries $y=0$ and $y=b$ radiating into a medium at 0^oC; boundaries $x=0$ and $x=a$ kept at 0^oC.

The procedure is entirely similar to that of Part III, Case 7. We put $v = v_1 + v_2 + v_3$ where

$$\lim_{t\to 0} v_1(x,y,z;t) = f(x,y,z)$$

$$\frac{\partial v_1}{\partial y} = 0 \quad \text{for} \quad y = 0. \text{ and } y = b$$

$$v_1 = 0 \quad \text{for} \quad x = 0 \text{ and } x = a$$

$$\lim_{t\to 0} v_2(x,y,z;t) = \lim_{t\to 0} v_3(x,y,z;t) = 0$$

$$\frac{\partial v_2}{\partial y} = h(v_1 + v_2) \quad \text{for} \quad y = 0$$

$$\frac{\partial v_1}{\partial y} = -h v_1 \quad \text{for} \quad y = b$$

$$\frac{\partial v_2}{\partial y} = h v_2 \quad \text{for} \quad y = 0$$

$$\frac{\partial v_3}{\partial y} = -h(v_1 + v_3) \quad \text{for} \quad y = b$$

$$v_2 = v_3 = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = a \,.$$

It is then readily seen that:

The solution v_1 is identical with that of Case 4.

The expression for v_2 may be obtained from the expression of u of Case 5 by replacing $\varphi(x,z;t)$ by $-v_1(x,0,z;t)$.

The expression for v_3 may be obtained from the expression of u of Case 6 by replacing $\varphi(x,z;t)$ by $-v_1(x,b,z;t)$.

. . . .

If in the "u" and "v" solutions obtained in the preceding cases we replace

$\sin \frac{n\pi x}{a}$ and $\sin \frac{n\pi \xi}{a}$ by $\cos \frac{n\pi x}{a}$ and $\cos \frac{n\pi \xi}{a}$ respectively,

we obtain the solutions appropriate to the cases where the boundaries x=0 and x=a are impervious to heat. Similarly if we replace $\sin \frac{n\pi x}{a}$ and

$\sin \frac{n\pi \xi}{a}$ by $\sin \frac{(2n+1)\pi x}{2a}$ and $\sin \frac{(2n+1)\pi \xi}{2a}$ respectively

we obtain the solutions appropriate to the cases where the boundary x=0 is kept at 0°C while the boundary x=a is impervious to heat.

Part VII. Heat Conduction in the Domain $D_3^{(v)}$

$$(0 \le x \le a, \quad 0 \le y \le b, \quad 0 \le z < \infty)$$

If in the solutions of the problems in Part VI we replace $\int_{-\infty}^{\infty} d\zeta$ by $\int_0^{\infty} d\zeta$ and the factor $e^{-\frac{(z-\zeta)^2}{4\kappa t}}$ by $e^{-\frac{(z-\zeta)^2}{4\kappa t}} \pm e^{-\frac{(z+\zeta)^2}{4\kappa t}}$

we obtain the solutions appropriate to the cases where the boundary z=0 is either impervious to heat or kept at 0°C.

It will therefore suffice to consider here problems involving radiation at the boundary z=0.

Case 1.

Boundary z=0 radiating into a medium at temperature $\varphi(x,y;t)$; other boundaries kept at 0°C.

Derivation of solution u(x,y,z;t)

For a function $\Phi(x,y)$ defined in the rectangle $0 \le x \le a$, $0 \le y \le b$ and vanishing on the sides of the rectangle we have the representation

$$\Phi(x,y) = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \quad \sin \frac{n\pi y}{b}$$

$$\cdot \int_0^a d\xi \int_0^b f(\xi,\eta) \sin \frac{m\pi\xi}{a} \cdot \sin \frac{n\pi\eta}{b} \, d\eta .$$

From the above identity, it follows that the Laplace transform of u(x,y,z;t) is

$$u^*(x,y,z;p) = \frac{4\lambda}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \quad \sin \frac{n\pi y}{b}$$

$$\cdot \int_0^a \sin \frac{m\pi\xi}{a} \, d\xi \int_0^b \frac{e^{-\gamma_{mn} z}}{\gamma_{mn} + h} \varphi^*(\xi,\eta;p) \sin \frac{n\pi\eta}{b} \, d\eta$$

where $\gamma_{mn} = \sqrt{\frac{p}{k} + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$.

From the last equation, it follows that

$$u(x,y,z;t) = \frac{4h}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b}$$

$$\cdot \int_0^a \sin \frac{m\pi\xi}{a} \, d\xi \int_0^b \sin \frac{n\pi\eta}{b} \, d\eta \int_0^t \varphi(\xi,\eta;t-\tau) \, \psi(z;\tau,\gamma_{mn}) \, d\tau$$

where $\psi(z;t,\gamma_{mn})$ is obtained from the inversion of

$$\int_0^\infty e^{-pt} \psi(z;t,\gamma_{mn}) \, dt = \frac{e^{-\gamma_{mn} z}}{\gamma_{mn} + h} .$$

By analogy with the expression for ψ in Part II, Case 3, we have

$$\psi(z;t,\gamma_{mn}) = \frac{e^{-\frac{k m^2 \pi^2 t}{a^2}} \cdot e^{-\frac{k n^2 \pi^2 t}{b^2}}}{2t\sqrt{\pi k t}} \cdot \int_0^\infty e^{-hp} (z+p) \, e^{-\frac{(z+p)^2}{4kt}} \, dp .$$

Substituting this expression of ψ in the formula for u(x,y,z;t) we obtain

$$u(x,y,z;t) = \frac{2h}{ab\sqrt{\pi k}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin\frac{n\pi x}{a} \quad \sin\frac{m\pi y}{b}$$

$$\cdot \int_0^a \sin\frac{n\pi\xi}{a}\,d\xi \int_0^b \sin\frac{m\pi\eta}{b}\,d\eta \int_0^t e^{-\frac{kn^2\pi^2\tau}{a^2}} \cdot e^{-\frac{km^2\pi^2\tau}{b^2}} \cdot \tau^{-\frac{1}{2}}$$

$$\cdot \varphi(\xi,\eta;t-\tau)\,d\tau \int_0^{\infty} e^{-hp}(z+p)\, e^{-\frac{(z+p)^2}{4k\tau}}\,dp \quad .$$

Derivation of solution v(x,y,z;t)

We put $v = v_1 + v_2$ where

$$\lim_{t\to 0} v_1(x,y,z;t) = f(x,y,z)$$

$$\frac{\partial v_1}{\partial z} = 0 \quad \text{for} \quad z = 0$$

$$\lim_{t\to 0} v_2(x,y,z;t) = 0$$

$$\frac{\partial v_2}{\partial z} = h(v_2 + v_1) \quad .$$

The solution v_1 may be obtained from the solution v of Part VI, Case 1 by replacing $\int_{-\infty}^{\infty} d\xi$ by $\int_0^{\infty} d\xi$ and the factor $e^{-\frac{(z-\zeta)^2}{4k t}}$ by $e^{-\frac{(z-\zeta)^2}{4kt}} + e^{-\frac{(z+\zeta)^2}{4kt}}$

The expression for v_2 may be obtained from that of u by replacing φ (x,y;t) by $-v_1(x,y,0;t)$.

If in the expression for u(x,y,z;t) and v(x,y,z;t) we replace sin $\frac{n\pi x}{a}$ and sin $\frac{n\pi\xi}{a}$ by cos $\frac{n\pi x}{a}$ and cos $\frac{n\pi\xi}{a}$ respectively, we obtain the solutions appropriate to the case where the boundaries x=0 and x=a are impervious to heat. Similarly if sin $\frac{n\pi x}{a}$ and sin $\frac{n\pi\xi}{a}$ are replaced by sin $\frac{(2n+1)\pi x}{2a}$ and $\frac{(2n+1)\pi\xi}{2a}$ we obtain the solutions appropriate to the case where the boundary x=0 is kept at 0° while the boundary x=a is impervious to heat. The expressions obtained from "u" solutions and "v" solutions by replacing sin $\frac{m\pi y}{b}$ and sin $\frac{m\pi\eta}{b}$ by cos $\frac{m\pi y}{b}$ and cos $\frac{m\pi\eta}{b}$ or sin $\frac{(2m+1)\pi y}{2b}$ and sin $\frac{(2m+1)\pi\eta}{2b}$ have similar meaning.

In conclusion it should be stated that the methods employed in this paper are not applicable to the case where radiation takes place at two bounding planes which are not parallel.

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